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# Three Dimensional Expansive Diffeomorphisms with Homoclinic Points

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**Abstract.** Let $M$ be a compact connected oriented three dimensional manifold and $f: M \to M$ an expansive diffeomorphism such that $\Omega(f) = M$. Let us also assume that there is a hyperbolic periodic point with a homoclinic intersection. Then $f$ is conjugate to an Anosov isomorphism of $T^3$. Moreover, we show that at a homoclinic point the stable and unstable manifolds of the hyperbolic periodic point are topologically transverse.

## 1. Introduction

Let $M$ be a compact metric space with metric dist: $M \times M \to \mathbb{R}$ defining its topology. Let $f: M \to M$ be a homeomorphism.

**Definition 1.1.** We say that $f$ is expansive if there exists a positive constant $\alpha$ such if we have $x, y \in M$ and for every $n \in \mathbb{Z}$ it holds that $\text{dist}(f^n(x), f^n(y)) \leq \alpha$ then $x = y$. The number $\alpha$ is called an expansivity constant for $f$.

Let us assume now that $M$ is a compact connected oriented $3D$-manifold and $f: M \to M$ an expansive diffeomorphism such that $\Omega(f) = M$. Let us assume that there is a hyperbolic periodic point with a homoclinic intersection associated to it. Then $f$ is conjugate to a linear Anosov diffeomorphism and $M \simeq T^3$, the three dimensional torus. Moreover, at any homoclinic point the stable and unstable manifolds of the hyperbolic periodic point must be topologically transverse.

In a previous paper [**Vi2**], the author proved that if we have an expansive homeomorphism $f$ defined on a three dimensional manifold $M$ with topologically hyperbolic periodic points dense in $M$ (see definitions

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at the end of this section), then $f$ is conjugate to an Anosov diffeomorphism and $M \simeq T^3$. In this article we show that it is enough to assume the existence of a local product structure given by local folliations invariant by $f$ defined in an open subset $A \subset M$ (see Theorem A), and we prove that this condition is fulfilled provided $f$ has a hyperbolic periodic point with a homoclinic intersection (see Theorem C). We also prove that the stable and unstable manifolds of the periodic point are topologically transverse at the homoclinic one (see Theorem B).

In fact we believe that it is enough to assume the hypotheses of $f$ to be expansive, $M$ to be three dimensional and $\Omega(f) = M$.

None of these hypotheses can be dropped. Pseudo-Anosov maps defined on closed surfaces of genus greater than one, are expansive with the non-wandering set all the surface and they are not conjugated to hyperbolic toral automorphisms. Moreover, for dimensions greater than three the device of taking products of pseudo-Anosov maps by Anosov ones allow us to construct expansive diffeomorphisms with the non-wandering set the whole manifold which are not conjugated to Anosov diffeomorphism.

In fact there are singular points in all the last mentioned examples in the sense that at these points the stable and unstable sets are not locally homeomorphic to Euclidean spaces, i.e.: they are not topological manifolds. Thus they cannot be conjugated to Anosov diffeomorphisms.

On the other hand, Franks and Robinson exhibited a counterexample in which $\Omega(f)$ is strictly included in $M = T^3 \neq T^3$. Moreover, $f$ is a quasi-Anosov diffeomorphism but it fails to be Anosov (see [**Fr-Ro**]).

In this paper we give results which represent a new approach to the above conjecture. We assume differentiability of $f$, in order to simplify the techniques although it seems that most of the results are valid for expansive homeomorphisms. And we assume that there is a hyperbolic periodic point with a homoclinic intersection. This assumption is perhaps easier to check than that of having a dense set of topologically hyperbolic periodic points.

Let us state the main results of this article, most of the definitions needed to understand their meaning are stated immediately below.

Let $M$ be a $C^\infty$ compact connected oriented three dimensional manifold and $f: M \to M$ a homeomorphism. Assume that $M$ is endowed with some metric dist: $M \times M \to \mathbb{R}^+$ defining its topology.

**Theorem A.** *Let $f: M \to M$ be an expansive homeomorphism (see definitions below in this section). Assume that the non wandering set of $f$, $\Omega(f)$, is $M$ and that there exists an open set $V \subset M$, $V \neq \varnothing$, such that there is a local product structure in $V$ given by local foliations invariant by $f$. Then $f$ is conjugate to a linear Anosov isomorphism and $M \simeq T^3$.*

**Theorem B.** *Let $f: M \to M$ be an expansive diffeomorphism with a hyperbolic periodic point $p$ with a homoclinic intersection $x \in W^s(\mathcal{O}(p)) \cap W^u(\mathcal{O}(p))$. Then $W^s(\mathcal{O}(p))$ is topologically transversal to $W^u(\mathcal{O}(p))$ at $x$.*

**Theorem C.** *Let $f: M \to M$ and $p$ be as in Theorem B. Assume that $\Omega(f) = M$. Then $M \simeq T^3$ and $f$ is conjugate to a linear Anosov isomorphism.*

Let us recall that $p \in M$ is $f$-periodic of prime period $k > 0$ if $f^k(p) = p$ and $f^j(p) \neq p$ if $j = 1, 2, \dots, k-1$. We define $Per(f) = p \in M/p$ is $f$-periodic.

We say that a linear isomorphism is hyperbolic if it has no eigenvalues of modulus 1.

**Definition 1.2.** Given $f$ a homeomorphism we say that a point $p \in M$ is an $f$ topologically hyperbolic periodic point (abbr.: $p$ is $f$-thp) if there is $k \in \mathbb{Z}^+$, $V(p)$ and $V(0)$ neighborhoods of $p$ and $0 \in \mathbb{R}^3$ respectively, a homeomorphism $h: V(p) \to V(0)$ and a hyperbolic linear isomorphism $T: \mathbb{R}^3 \to \mathbb{R}^3$ such that $f^k(p) = p$ and $h^{-1} \circ T \circ h_{|V(p)} = f^k_{|V(p)}$. If $s$ is the number of eigenvalues of $T$ of modulus less than 1 we say that $p$ is of index $s$.

We define $Per_H(f) = p \in M/p$ is $f$-thp. Observe that if $f$ is a diffeomorphism and $p$ is a hyperbolic periodic point in the usual sense, the Hartman-Groβman Theorem says that it is an $f$-thp.

**Definition 1.3.** For $x \in M$ we define

$$W^s_\epsilon(x, f) = \{y \in M / \operatorname{dist}(f^k(x), f^k(y)) \leq \epsilon; k \geq 0\}$$

as the local $\epsilon$-stable set for the point $x$ and the homeomorphism $f$.

Analogously we define the local $\epsilon$-unstable set for $x$ and $f$ as $W^u_\epsilon(x,f) = W^s_\epsilon(x,f^{-1}\}$. If there is no ambiguity we shall usually omit any reference to $\epsilon$ and $f$ and speak about local stable and unstable sets of the point $x$ denoting them by $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ respectively. Observe that $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ verify that $f(W^s_\epsilon(x)) \subset W^s_\epsilon(f(x))$ and $f^{-1}(W^u_\epsilon(x)) \subset W^u_\epsilon(f^{-1}(x))$.

**Definition 1.4.** We define

$$W^s(x,f) = \{y \in M / \lim_{k\to+\infty} \operatorname{dist}(f^k(x), f^k(y)) = 0\}$$

as the stable set for the point $x$ and the homeomorphism $f$.

Analogously we define the unstable set for $x$, $f$ as $W^u(x,f) = W^s(x,f^{-1})$. We usually will omit the reference to $f$. If $p$ is a periodic point with prime period $k$ we define the stable set of the $f$-orbit through $p$, denoted as $\mathcal{O}(p)$, as

$$W^s(\mathcal{O}(p),f) = \bigcup_{j=0}^{k-1} W^s(f^j(p),f)$$

We put $W^u(\mathcal{O}(p),f) = W^s(\mathcal{O}(p),f^{-1})$. Again we remark that the reference to $f$ will be avoided if no ambiguity results.

**Definition 1.5.** Given a set $A$ in a metric space $(X,d)$ we define its $\delta$-parallel body

$$[A]_\delta = \{y \in X / \inf_{x\in A} d(y,x) \leq \delta\}.$$

The Hausdorff distance $H$ dist between two non empty compact sets $A$, $B \subset X$ is

$$H\operatorname{dist}(A,B) = \inf\{\delta \geq 0 / A \subset [B]_\delta \quad \text{and} \quad B \subset [A]_\delta\}.$$

With this distance the space $\mathcal{C} = \{C \in M / C \neq \varnothing \quad \text{is compact}\}$ is a complete space (see [**Fa**]).

**Definition 1.6.** We say that a sequence of compacta $\{C_n\}$ converges in the Hausdorff metric to $C$ and write $H \lim C_n = C$ if given $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $H\operatorname{dist}(C_n, C) < \epsilon$.

**Definition 1.7.** Let $S$ be a surface and $\gamma$ a curve both immersed in $M$. We say that $S$ is topologically transversal to $\gamma$ at a point $x \in S \cap \gamma$ or that $S$ and $\gamma$ are topologically transverse at $x$, if there is a neighbourhood $V$ of $x$ and a homeomorphism $h: V \to V(O) \subset \mathbb{R}^3$ such that $h(S \cap V) = Oxy \cap V(O)$ and $h(\gamma \cap V) = Oz \cap V(O)$.

As usual $Oxy = \{x, y, z) \in \mathbb{R}^3/z = 0\}$ and $Oz = \{(x, y, z) \in \mathbb{R}^3/x = y = 0\}$.

**Definition 1.8.** Given $f: M \to M$ and a point $x \in M$ we say that there is a local product structure in $x$ (abbr.: $x$ has an $f$-lps) if there is a neighbourhood $V(x)$, and a homeomorphism $h: V(x) \to [-1, 1]^3$, the standard $3D$ cube, such that either $h$ sends the local stable sets of points in $V(x)$ onto horizontal squares and the unstable sets onto vertical segments or, alternatively, sends the local stable sets of points in $V(x)$ onto vertical lines and the unstable sets onto horizontal squares.

Observe that the local product structure is invariant by $f$, i.e.: if there is a local product structure for $x$ then there is a local product structure for $f^k(x)$ for $k \in \mathbb{Z}$. Our definition of local product structure differs from the usual one (see [**Sh**]). We require the existence of local basic foliations. We say that a set $A \subset M$ has an $f$-local product structure (abbr. $A$ has an $f$-lps) if every point of $A$ has an $f$-lps in $M$. By definition $A$ is open. The open ball of center $x$ and radius $r > 0$ is the set $B(x, r) = \{y \in M/ \operatorname{dist}(x, y) < r\}$. As $M$ is a compact manifold there is $r_1 > 0$ such that for all $0 < r < r_1 B(x, r)$ is homeomorphic to $B^3$ the standard 3-cell of $\mathbb{R}^3$. We assume from now on that we take $r \leq r_1$.

**Definition 1.9.** Let $D$, $D' \subset M$ be continua such that both separates $B(x, r)$ and such that $D$ is contained in one region of $B(x, r) \backslash D'$ and $D'$ is contained in one region of $B(x, r) \backslash D$. We say that a point $z$ is between $D$ and $D'$ in $B(x, r)$ if it belongs to the region of $B(x, r') \backslash D$ containing $D'$ intersected with that of $B(x, r') \backslash D'$ containing $D$. We say that a set $S \subset B(x, r)$ is between $D$ and $D'$ in $B(x, r)$ if every point in $S$ is between $D$ and $D'$.

Throughout this paper, except in section 2, we assume that $M$ is a compact connected orientable smooth $3D$-manifold so we will usually avoid to state it.

## 2. Basic properties of expansive maps

In this section we state most of the properties of expansive homeomorphisms used in the subsequent ones. For the proofs of such properties we will usually refer the reader to the corresponding reference. Nevertheless in certain cases we will sketch the proofs.

**Proposition 2.1.** **(Hyperbolic metric.)** *Let $(M,d)$ be a compact metric space and $f\colon M \to M$ an expansive homeomorphism with constant of expansivity $\alpha > 0$. Then there are $\lambda$, $0 < \lambda < 1$ and an adapted metric $\delta$, defining the same topology as $d$, such that for all $x \in M$ and for all $y \in W^s_\epsilon(x)$ it holds that $\delta(f(x), f(y)) < \lambda\delta(x,y)$. The same is true for $y \in W^u_\epsilon(x)$ changing $f$ by $f^{-1}$. Moreover, there is $\mu > 0$ such that for all $x$, $y \in M$ we have*

$$\max(\delta(f(x), f(y)), \delta(f^{-1}(x), f^{-1}(y))) \geq \max(\lambda\delta(x,y), \mu)$$

**Proof.** See [**Re**] Theorem 1 and [**Ft**], §5, Theorem 5.1. □

In propositions 2.2, 2.4, 2.6, 2.7, 2.8 and 2.10 let $M$ be a compact $n$-dimensional manifold and $f$ an expansive homeomorphism of $M$ with expansivity constant $\alpha > 0$.

**Proposition 2.2.** **(Existence of Lyapunov functions.)** *There are functions $U$, $V$ and $W$ defined in a neighbourhood $N$ of the diagonal $M \times M, U, V, W\colon N \to \mathbb{R}$ such that $V(x,y) = \Delta U(x,y) = U(f(x), f(y)) - U(x,y)$ and $W(x,y) = \Delta V(x,y) = V(f(x), f(y)) - V(x,y) = \Delta(\Delta U) = U(f^2(x), f^2(y)) - 2U(f(x), f(y)) + U(x,y)$ with the properties that $U(x,y)$ and $W(x,y)$ vanish only at the diagonal of $M \times M$ and are positive elsewhere, and $V(x,y) > 0$ if $y \in W^u_\epsilon(x)$ and $V(x,y) < 0$ if $y \in W^s_\epsilon(x), y \neq x$, where $\epsilon > 0$ is less than some expansivity constant $\alpha$.*

**Proof.** See [**Le3**], §1, see also [**Le1**] §4. □

**Remark 2.3.** As in [**Le1**] §4 we may construct Lyapunov functions for the suspension flow $\phi$ associated to $f$.

Let $(\tilde{M}, \phi)$ be the suspension of $(M, f)$ under the constant function 1. Identify $M$ with $\pi(M \times \{0\})$, $\pi$ being the suspension projection of $M \times \mathbb{R}$ onto $\tilde{M}$, and $f^n(y)$ with $\phi(y, n)$. Call $M_t$ the manifold, $\phi(M, t)$. Let dist be some Riemannian metric dist: $M \times M \to \mathbb{R}$. The expansivity of $f$ implies that there is $\beta > 0$ such that if $y,\ z \in M$ and for all $s \in \mathbb{R}\,\mathrm{dist}(\phi(y,t), \phi(z,t)) \leq \beta$ then $y = z$; observe that we don't loose generality supposing that $\alpha = \beta$. Then there are continuous functions $U, U', U'', : N \to \mathbb{R}, N = \{(x,y) \in \tilde{M} \times \tilde{M} / \exists t \in \mathbb{R} : x, y \in M_t \quad \text{and} \quad \mathrm{dist}(x,y) \leq \lambda\}, \lambda > 0$, such that:

$$U(x,y) \geq 0 \quad \text{and} \quad U(x,y) = 0 \quad \text{iff} \quad x = y$$

$$U'(x,y) = \lim_{t \to 0} \frac{U(\phi(x,t), \phi(y,t)) - U(x,y)}{t}$$

and if $y \in W^s_\epsilon(x) (y \in W^u_\epsilon(x)), y \neq x$, then $U'(x,y) < 0$ (resp. $U'(x,y) > 0$).

$$U''(x,y) = \lim_{t \to 0} \frac{U'(\phi(x,t), \phi(y,t)) - U'(x,y)}{t} \quad \text{and} \quad U''(x,y) > 0, x \neq y.$$

We will have the opportunity to use Lyapunov functions for the suspension flow $\phi$ of $f$ below in sections 4 and 5.

**Definition 2.1.** Let $M$ and $f$ be as above. We say that a point $x \in M$ is a stable point for $f$ (Lyapunov stable) if for every $\epsilon > 0$ there is $\delta > 0$ such that $\mathrm{dist}(x,y) < \delta$ implies $\mathrm{dist}(f^n(x), f^n(y)) < \epsilon$ for all $n > 0$.

**Proposition 2.4.** *If $f : M \to M$ is expansive then it has no stable points.*

**Proof.** See [**Le2**] lemma 2.7. □

**Corollary 2.5.** *If $p$ is $f$-*thp *and $f$ is expansive then $p$ must be either of index* 1 *or* 2.

**Proof.** Otherwise we will have a periodic repeller or attractor and therefore a Lyapunov stable point either for $f$ or $f^{-1}$. □

Proposition 2.4 is used in [**Le3**] to prove the following result.

**Proposition 2.6.** *Let $0 < \epsilon < \alpha$. There is $r, 0 < r \leq \epsilon$ such that for every $x \in M$ there exists a compact connected set $D(x) \subset W^s_\epsilon(x)(C(x) \subset$*

*$W^u_\epsilon(x)$) such that $x \in D(x)$ (resp.: $x \in C(x)$) and for all open set $A \subset B(x,r), x \in A$, we have that $D(x) \cap \partial A \neq \varnothing$ (resp.: $C(x) \cap \partial A \neq \varnothing$).*

**Proof.** In [**Le3**], lemma 2.1, it is proved that given $A \subset M$ an open set, $x \in A \subset B(x,r)$ there exists a compact connected set $D(x), x \in D(x) \subset clos(A), D(x) \cap \partial A \neq \varnothing$, such that for $y \in D(x), \operatorname{dist}(f^n(x), f^n(y)) \leq \epsilon$ if $n \geq 0$. It seems from this that $D(x)$ depends on $A$. On the other hand, it is not difficult to see that such a $D(x)$ may be constructed for $B(x,r)$ itself, and therefore it will cut the boundary of any open set $A \subset B(x,r)$ such that $x \in A$. □

**Proposition 2.7.** *Let $0 < \sigma < \epsilon$. There is $r > 0$ such that if $y \in W^s_\epsilon(x)$ and $\operatorname{dist}(x,y) < r$ then $y \in W^s_\sigma(x)$.*

**Proof.** See [**Le3**], lemma 2.2. □

**Proposition 2.8.** *Let $\alpha, \epsilon$ and $r$ be as in Proposition 2.6. Given $0 < r' < r$ there exist $\mu > 0$ and $\lambda > 0$ such that if $\operatorname{dist}(x,y) < \lambda$ then $\operatorname{dist}(W^s_\epsilon(x) \cap \partial B(x,r'), W^u_\epsilon(y) \cap \partial B(x,r') > \mu$. (We put $\operatorname{dist}(A,B) = \inf\{\operatorname{dist}(x,y)/x \in A, y \in B\}$.)*

**Proof.** Assume that the contrary holds. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ and a number $r'$, $r > r' > 0$ such that

$$\operatorname{dist}(x_n, y_n) < \frac{1}{n}$$

and

$$\operatorname{dist}(W^s_\epsilon(x_n) \cap \partial B(x_n, r'), W^u_\epsilon(y_n) \cap \partial B(x_n, r')) \leq \frac{1}{n}.$$

Since $M$ is compact we may assume that $\{x_n\}$ converges to a point $x$. Also $\lim y_n = x$. As $r' < r$ we have that $W^s_\epsilon(x_n) \cap \partial B(x_n, r') \neq \varnothing$ and $W_\epsilon(y_n) \cap \partial B(x_n, r') \neq \varnothing$. Letting $z_n \in W^s_\epsilon(x_n) \cap \partial B(x_n, r')$ and $w_n \in W^u_\epsilon(y_n) \cap \partial B(x_n, r')$ be such that $\operatorname{dist}(z_n, w_n) \leq \frac{3}{n}$ and taking a convergent subsequence $\{z_{n_k}\}$ from $\{z_n\}$ we will have that $\{w_{n_k}\}$ has the same limit $z \in \partial B(x, r')$. But $z \in W^s_\epsilon(x) \cap W^u_\epsilon(x)$: given $h \in \mathbb{Z}^+$ we have that $\operatorname{dist}(f^h(x_{n_k}\}, f^h(z_{n_k}\} \leq \epsilon$ so that letting $k \to +\infty$ we have that $\operatorname{dist}(f^h(x), f^h(z)) \leq \epsilon$ hence $z \in W^s_\epsilon(x)$; analogously $z \in W^u_\epsilon(x)$. Thus $z \in W^s_\epsilon(x) \cap W^u_\epsilon(x) \cap \partial B(x, r')$ so $x \neq z$ and $\operatorname{dist}(f^n(x), f^n(z)) < \alpha \ \forall n \in \mathbb{Z}$ contradicting the expansivity of $f$. □

**Remark 2.9.** The same kind of arguments enable us to prove that there exist $\lambda > 0$ and $\mu > 0$ such that if $\text{dist}(x,y) < \lambda$ then

$$\text{dist}(W^s_\epsilon(x)\backslash B(x,r'), W^u_\epsilon(y)\backslash B(x,r')) > \mu$$

**Proposition 2.10.** *Let $c > 0$ be an expansivity constant for $f$. Then for all $\epsilon > 0$ there is $N > 0$ such that $f^n(W^s_c(x)) \subset W^s_\epsilon(f^n(x))$ and $f^{-n}(W^u_c(x)) \subset W^u_\epsilon(f^{-n}(x))$ for all $n \geq N$ and $x \in M$.*

**Proof.** See [**Ma**], Lemma I p. 315. □

From now on let us assume that $M$ is three dimensional, hence, by 2.10, if $p$ is an $f$-thp point of index 2 then for $0 < \epsilon < \alpha, W^s_\epsilon(p)$ is a 2-disk and $W^u_\epsilon(p)$ is an arc.

**Proposition 2.11.** *If $p$ is an $f$-*thp *point of index 2 then given $\epsilon, 0 < \epsilon < \frac{\alpha}{2}$, there is $r > 0$, depending only on $\epsilon$, such that*

1. *$S(p)$, the connected component of $W^s_\epsilon(p) \cap B(p, r_0)$ containing $p$, separates $B(p, r_0)$ in two connected components.*
2. *$U(p)$, the connected component of $W^u_\epsilon(p) \cap B(p, r_0)$ containing $p$, is an arc which reaches the boundary of $B(p, r)$, for all $0 < r < r_0$, in both components of $B(p, r_0)\backslash S(p)$.*

**Proof.** The proof of this proposition may be found in [**Vi1**] section 2. □

Let $S'(x, y, r, f)$ = (connected component of $clos(B, x, r)) \cap W^s_\epsilon(y)$ containing $y$) and

$$S(x,y,r) = \varprojlim_{0<r'<r} S'(x,y,r',f)$$

($S(x, y, r)$ is the inverse limit in $r'$ of $S'(x, y, r', f)$). We define $U'(x, y, r, f)$ as $S'(x, y, r, f^{-1})$ and

$$U(x,y,r) = \varprojlim_{0<r'<r} U'(x,y,r',f)$$

We have that $S(x, y, r)$ and $U(x, y, r)$ are compact sets. When $x = y$ we denote $S(x, x, r)$ as $S(x, r)$ and $U(x, x, r)$ as $U(x, r)$. It is not difficult to prove from 2.11 that $S(x, r)$ separates $B(x, r)$ for $0 < r \leq r_0$ in two connected components and that $U(x, r)$ is an arc which reaches the boundary of $B(x, r)$ in both components.

**Remark 2.12.** It also occurs that if $x$ is close enough to $p$ then $S(x,p,r)$ separates $B(x,r)$ in two connected components and $U(x,p,r)$ is an arc reaching $\partial B(x,r)$ in both components.

In what follows we choose $\epsilon < \frac{\alpha}{2}$ and, (Proposition 2.7) $r > 0$ such that $r < r_0$ and such that if $y \in W^t_{2\epsilon}(x)$ and $\text{dist}(x,y) \leq r$ then $y \in W^t_\epsilon(x), t = s, u$.

**Proposition 2.13.** *Let $A$ be an open set in which there is an $f$-lps and $x \in \partial A, x = \lim_{n\to+\infty} p_n$ where $\{p_n\} \subset Per_H(f) \cap A$ and let the positive real numbers $\epsilon$ and $r \leq r_0$ as above.*

*Then there is $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ index $(p_n)$ = constant = 1 or 2.*

*Assuming that it is 2, there is a subsequence $\{p_{n_k}\} \subset \{p_n\}$ and there are continua $C(x) \subset W^u_\epsilon(x)$ and $D(x) \subset W^s_\epsilon(x)$ such that:*

1. *$C(x) = H\lim_{k\to+\infty}(U(p_{n_k}, r))$ and $D(x) = H\lim_{k\to+\infty}(S(p_{n_k}, r))$.*
2. *$D(x)$ separates $B(x,r)$ and $C(x)$ has points in two connected components of $B(x,r)\backslash D(x)$ and in both components $C(x)$ reaches $\partial B(x,r)$.*
3. *There exists (at least) a component $X$ of $B(x,r)\backslash D(x)$ in which there are infinitely many points of $\{p_{n_k}\}$. If $\{p'_n\} \subset Per_H(f) \cap X$ and $x = \lim_{n\to+\infty} p'_n$ then $D(x) = H\lim_{n\to+\infty} S(p'_n, r)$.*

**Proof.** As $p_n \in Per_H(f) \subset A$ there is an $f$-lps in a neighbourhood $V$ of $p_n$. Hence, given $y \in V$, there is $c > 0$ such that $W^s_c(y)$ and $W^u_c(y)$ are topological manifolds. By Proposition 2.10, for all $y \in M$, for all $\epsilon, 0 < \epsilon < \alpha$, there is $N \in \mathbb{N}$ such that if $k \geq N f^k(W^s_\epsilon(y)) \subset W^s_c(f^k(y))$ and $f^{-k}(W^u_\epsilon(y)) \subset W^u_c(f^{-k}(y))$, therefore $W^s_\epsilon(p_n)$ and $W^u_\epsilon(p_n)$ are topological manifolds of the same dimension of that of $W^s_c(p_n)$ and $W^u_c(p_n)$ respectively. As $x = \lim_{n\to+\infty} p_n$, for every $\lambda > 0$ there is $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $h \in \mathbb{N}$, $\text{dist}(p_n, p_{n+h}) < \lambda$. From this it may be proved that $\dim(W^s_\epsilon(p_n)) = \dim(W^s_\epsilon(p_{n+h}))$ if we choose $\lambda > 0$ small enough, otherwise, if $W^s_\epsilon(p_n)$ and $W^u_\epsilon(p_{n+h}))$ are both two dimensional, then, by Proposition 2.11, they will have more than one point of intersection which contradicts expansivity (see [**Vi1**] §3 for more details).

The proofs of 1., 2. and 3. may be found in [**Vi2**], nevertheless we will sketch how to construct $D(x)$ and $C(x)$ for the sake of completeness.

**1.** As $\mathcal{C} = \{C \subset M / C \text{is compact}\}$ with the metric $H\,\text{dist}$ is a complete space and $\{S(p_n, r)\}$ is bounded we have that there is $\{p_{n_k}\}$ such that $H\lim(S(p_{n_k}, r)$ exists. Let us call $D(x)$ to this limit. The sequence $\{p_n\}$ converges to $x$, so $x \in D(x)$. To see that $D(x) \in W^s_\epsilon(x)$ let $y$ be any point of $D(x)$, then there is $y_{n_k} \in S(p_{n_k}, r)$ such that $\lim_{k\to+\infty} y_{n_k} = y$. Given $h \geq 0$, as $y_{n_k} \in S(p_{n_k}, r) \subset W^s_\epsilon(p_{n_k})$ we have that $\text{dist}(f^h(y_{n_k}), f^h(p_{n_k})) \leq \epsilon$, taking limits with $k \to +\infty$ we obtain $\text{dist}(f^h(y), f^h(x)) \leq \epsilon$ so that $y \in W^s_\epsilon(x)$. Therefore $D(x) \subset W^s_\epsilon(x)$. Take from $U(p_{n_k}, r)$ a convergent subsequence and call $C(x)$ to its limit. As above we may prove that $C(x) \subset W^u_\epsilon(x)$ and changing notation, if it were necessary, we may suppose that $U(p_{n_k}, r)$ itself converges proving 1.

**2.** The proof of 2 uses the fact that $S(p_{n_k}, r)$ separates $B(x, r)$ if $p_{n_k}$ is close enough to $x$ and that as $\lim_{k\to+\infty} p_{n_k} = x$, $S(p_{n_k}, r)$ cannot shrink to a point (it joins $p_{n_k}$ with $\partial B(x, r)$) and cannot shrink to a continuous of dimension less than 2, if not, points of $S(p_{n_k}, r)$ will collapse with points of $U(p_{n_k}, r)$ in the boundary of $B(x, r)$ contradicting expansiveness.

**3.** To prove 3 we use that there is $K \in \mathbb{N}$ such that for $k \geq K S(p_{n_k}, r)$ separates $C(x)$ and therefore they have a point $y_{n_k}$ of intersection (unique by the expansive properties of $f$). One of the subcontinua $C^+(x)$ or $C^-(x)$ in which $D(x)$ separates $C(x)$ must intersect $S(p_{n_k}, r)$ for infinitely many values of $k$. Hence, for these $k$, $p_{n_k}$ and $y_k$ are in the same connected component of $B(x, r) \backslash D(x)$. Otherwise $p_{n_k}$ and $y_{n_k}$ are separated by $D(x)$, so that $S(p_{n_k}, r)$ will intersect $D(x)$ for infinitely many values of $k$. Hence there are two different periodic points $p_{n_k}$ and $p_{n_j}$ in $W^s_{2\epsilon}(x)$ and therefore, by our choice of $r$, $p_{n_k}$ and $p_{n_j}$ are in $W^s_\epsilon(x)$. Then $p_{n_k} \in W^s_{2\epsilon}(p_{n_j})$ and, as $2\epsilon < \alpha$, this implies the absurd that $lim_{n\to+\infty} f^{hn}(p_{n_k}) = p_{n_j}$, where $h$ is the period of $p_{n_j}$. Therefore there exists a component of $B(x, r) \backslash D(x)$ with infinitely many points of $\{p_{n_k}\}$. Let us call $X$ to the connected component with infinitely many points of $\{p_{n_k}\}$ (if there are more than one component with this property choose one of them). To prove that if $\{p'_n\} \subset Per_H(f) \cap X$ and $x = \lim p'_n$ then $D(x) = H\lim S(p'_n, r)$ see [**Vi2**]. □

The following proposition states the principal result of [**Vi2**].

**Proposition 2.14.** *The following statements are equivalent:*

a) *$f$ is conjugate to a linear Anosov isomorphism.*

b) *$f$ is expansive and $clos(Per_H(f)) = M$.*

**Proof.** a) implies b) follows from well known facts about linear Anosov diffeomorphisms and the commutativity of the diagram,

$$\begin{array}{ccc} T^3 & \xrightarrow{a} & T3 \\ h\big\downarrow & & \big\uparrow h \\ M & \xrightarrow{f} & M \end{array}$$

where $A$ is Anosov and $h$ is a homeomorphism, (see [**Fr**]). To see that b) implies a) see [**Vi2**], sections 9 and 10. □

**Proposition 2.15.** *If $f\colon M \to M$ is an expansive homeomorphism with $\Omega(f) = \mathcal{M}$ and there is an $f$-lps in an open set $A \subset M$ then the periodic points are dense in $A$.*

**Proof.** Observe that if $\epsilon > 0$ is less than $\frac{\alpha}{2}$, $\alpha$ an expansivity constant for $f$, and $V$ is a connected neighborhood in which there is an $f$-lps and there is $x_0 \in V$ such that $\dim(W^s_\epsilon(x_0)) = 2$ (hence $\dim(W^u_\epsilon(x_0) = 1)$ then $\dim(W^s_\epsilon(x)) = 2$ for all $x \in V$. If it were not true, we would have points $x_0$ and $x_1$ in $V$ such that $\dim(W^s_\epsilon(x_0)) = \dim(W^u_\epsilon(x_1) = 2$, and we may assume by the connectedness of $V$ that $x_1$ is in the product neighborhood $V(x_0) \simeq [0,1]^3$. Therefore $W^u_\epsilon(x_1) \cap W^s_\epsilon(x_0)$ will contain an arc of points such that if $x, y$ are in this arc then $\forall n \in \mathbb{Z}\, \mathrm{dist}(f^n(x), f^n(y)) < \alpha$. But this contradicts the expansivity of $f$. In order to prove that periodic points are dense in $A$ we use arguments similar to those of Franks (see [**Fr**] lemma 1.7). In fact, having an $f$-lps and in view of Proposition 2.1, the hyperbolic theory applies.

Let $V \subset A$ be a connected neighborhood of $x \in A$ and let us assume, by the above arguments, that $\dim(W^s_\epsilon(y)) = 2$ and $\dim(W^u_\epsilon(y)) = 1$ for all $y \in V$. Let $N$ be a product neighborhood $N \approx W^s_\epsilon(x) \times W^u_\epsilon(x)$ with $\epsilon$ so small that $N \subset V$ and let $N' \approx W^s_{\epsilon/4}(x) \times W^u_{\epsilon/4}(x)$. Since $\Omega(f) = M$, there exists $x'$ such that $W^u_{\epsilon/8}(x') \subset N'$ and $f^n(x') \in N'$ for some $n$

which is sufficiently large that $L = f^n(W^u_{\epsilon/8}(x')) \cap N \supset W^u_{\epsilon/2}(f^n(x'))$. By 2.10, such $n$ exists independently of the particular $x'$. Let $L_0$ be the component of $L$ containing $f^n(x')$. Then, by the existence of the $f$-lps, $L_0$ is an arc. We define a map $h: L_0 \to L_0$ in this way: let $z \in L_0$, $z' \in W^u_{\epsilon/8}(x')$ be such that $z' = f^{-n}(z)$. As there is an $f$-lps in $N'W^s_{\epsilon/4}(z')$ intersects $L_0$ at, say, $w$. Define $w = h(z)$. $h$ is a continuous map from an arc into itself so it has a fixed point $y$. Let $D = W^s_{\epsilon/2}(f^{-n}(y))$, then $D$ is a 2-disk and we may assume that $n$ is large enough to guarantee that $f^n(W^s_{\epsilon/2}(f^{-n}(y))) \subset W^s_{\epsilon/8}(y)$. But $y$ is in $W^s_{\epsilon/4}(f^{-n}(y))$ so $f^n(W^s_{\epsilon/2}(f^{-n}(y))) \subset W^s_{\epsilon/2}(f^{-n}(y)) = D$. Hence $f^n$ is continuous and has a fixed point $p \in N$. It is clear that $f^n(p) = p$. □

**Proposition 2.16.** *If $p$ is a periodic point in an open set $A \subset M$ such that there is an $f$-*lps *in $A$ then $p$ is an $f$-*thp *point.*

**Proof.** Assume, without loss of generality that $\dim(W^s_\epsilon(p)) = 2$, $\dim(W^u_\epsilon(p)) = 1$. Moreover, as there is an $f$-lps in $p$, we may also assume that $p = 0$, the origin of $\mathbb{R}^3$, and that

$$W^s_\epsilon(p) \supset D = \{(x,y,z) \in \mathbb{R}^3/z = 0 \quad \text{and} \quad x^2 + y^2 \le 1\}$$

and

$$W^u_\epsilon(p) \supset I = \{(x,y,z) \in \mathbb{R}^3/x = y = 0 \quad \text{and} \quad -1 \le z \le 1\}.$$

Take $k > 0$ such that $f^k(p) = p$ and $f^k_{|W^s_\epsilon(p)}$ and $f^k(p) = p$ and $f^k_{|W^u_\epsilon(p)}$ are orientation preserving and such that $f^k(D) \subset \text{int}(D)$ and $f^{-k}(I) \subset \text{int}(I)$. Let $U$ be the unbounded component of $Oxy \backslash f^k(\partial D)$. The intersection of $clos(U)$ with $D$ is an annulus $A$. By the Schoenflies Theorem (see [**Mo**] Ch. 10 and Ch. 13 Th. 1) $f^k(D)$ is a 2-disk homeomorphic to $\{(x,y)\mathbb{R}^2/x^2 + y^2 \le 1/4\}$ by a homeomorphism $\psi: \mathbb{R}^2 \to \mathbb{R}^2$ which is the identity outside $\{(x,y) \in \mathbb{R}^2/x^2 + y^2 \le 1/2\}$ and such that if $x \in \partial D$ then $\psi(f^k(x)) = \frac{1}{2}x$. We foliate the annulus $\mathcal{A}_0 = \{(x,y) \in \mathbb{R}^2/1/4 \le x^2 + y^2 \le 1\}$ with two foliations $\mathcal{F}_0$ and $\mathcal{G}_0$, transverse to $\mathcal{F}_0$, such that the leaves of $\mathcal{F}_0$ are the circles $x^2 + y^2 = r^2$, $1/2 \le r \le 1$, and the leaves of $\mathcal{G}_0$ are constructed with segments joining $\psi(x) = x \in \partial D$ with $\psi(f^k(x)) = \frac{1}{2}x$. By using $\mathcal{F}_0$ and $\mathcal{G}_0$ we induce foliations $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{A}$ via $\psi^{-1}$; i.e.: $\mathcal{F} = \psi^{-1}\mathcal{F}_0$ and $\mathcal{G} = \psi^{-1}\mathcal{G}_0$. This construction allows us to define a $C^0$ flow $\Phi$ on $D$ using the device of defining it first in $\partial D$ by

$$\Phi_t(x) = \psi^{-1}(\frac{1}{t+1}\psi(x)), \quad 0 \leq t \leq 1$$

and for $t \in [n, n+1)$ we put $\Phi_t(x) = f^{kn}(\Phi_{t-n}(x))$. If we now have a point $y$ in $D\backslash\{O\}$, there is $j \in \mathbb{N}$ such that $f^{-j}(y) \in \mathcal{A}$, and there is $\tau \in [0,1]$ such that $x = \Phi_\tau(f^{-j}(y)) \in \partial D$. We put $\Phi_t(y) = \Phi_{j+\tau+t}(x)$. Finally we define $\Phi_t(O) = O$ for all $t \geq 0$. It is clear that if we call $T_{1/2}$ to $\Phi_1$ then $f^k(y) = \psi^{-1} \circ T_{1/2} \circ \psi(y), T_{1/2}(y) = \frac{1}{2}y$. In an analogous way we may find a homeomorphism $\xi: I \to \mathbb{R}$ such that $f^k(x) = \xi^{-1} \circ T_2 \circ \xi(x), T_2(x) = 2x$. Let us find a suitable neighborhood $N$ of $O$ such that $f^{-k}(N)$, $f^k(N) \subset D \times I$. If we have that $z \in N$ and $x = W^s_\epsilon(z) \cap I$, $y = W^u_\epsilon(z) \cap D$, then, by the invariance of the $f$-lps, we have that $f^k(z) = W^u_\epsilon(f^k(y)) \cap W^s_\epsilon(f^k(x))$ and therefore, defining $F(y,z) = (\psi(y), \xi(x))$ and $T(y,x) = (T_{1/2}(y), T_2(x))$, we have that $f^k(z) = F^{-1} \circ T \circ F(z)$. Thus we have proved that $p$ is an $f$-thp point. □

**Corollary 2.17.** *If $\Omega(f) = M$ and there is an open $f$-invariant dense set $A \subset M$ with an $f$-lps then $f$ is conjugate to an Anosov diffeomorphism.*

**Proof.** With the above hypotheses, by 2.15 and 2.16, we have that $Per_H(f)$ is dense in $M$ so the thesis follows from 2.14.

**Remark 2.18.** The above corollary, as well as Proposition 2.14, are true only in the three dimensional case. PseudoAnosov maps defined in a surface of genus greater than one verify the hypothesis but certainly are not conjugate to Anosov diffeomorphisms. By taking products of pseudoAnosov maps by Anosov ones we may see that for dimensions greater than three the statement is false too. On the other hand, there are no expansive homeomorphisms in the one dimensional case.

## 3. Proof of Theorem A

Let $A \subset M$ be the maximal open set in which ther is an $f$-lps. It is easy to see that $A$ is invariant. In view of 2.17, to prove Theorem A, it suffices to show that $A$ is dense in $M$. Let us suppose that this is false. Therefore there exists an open set $B \subset M$, $B \neq \varnothing$, such that nor in $B$ neither in any non empty open subset of $B$ is defined an $f$-lps. We may assume that $B$ is connected. By hypothesis $B \neq M$ hence $\partial B \neq \varnothing$. Let $x \in \partial B$, then $x$ also belongs to the boundary of $A$. By 2.15 and 2.16 there is a sequence $\{p_n\} \subset Per_H(f) \cap A$ such that $\lim p_n = x$. By 2.13 we may assume that $\{S(p_n, r)\}$ and $\{U(p_n, r)\}$ are convergent sequences to $D(x) \subset W^s_\epsilon(x)$ and $C(x) \subset W^u_\epsilon(x)$.

**Remark 3.1.** If we have an $f$-thp point $p \in A$ then we have an $f$-lps in an open connected neighborhood $N$ of $U(p,r) \cup S(p,r)$. Thus $A$ includes $U(p,r) \cup S(p,r)$.

Before going into the details let us give a brief idea of the proof of Theorem A, see also figure 1.

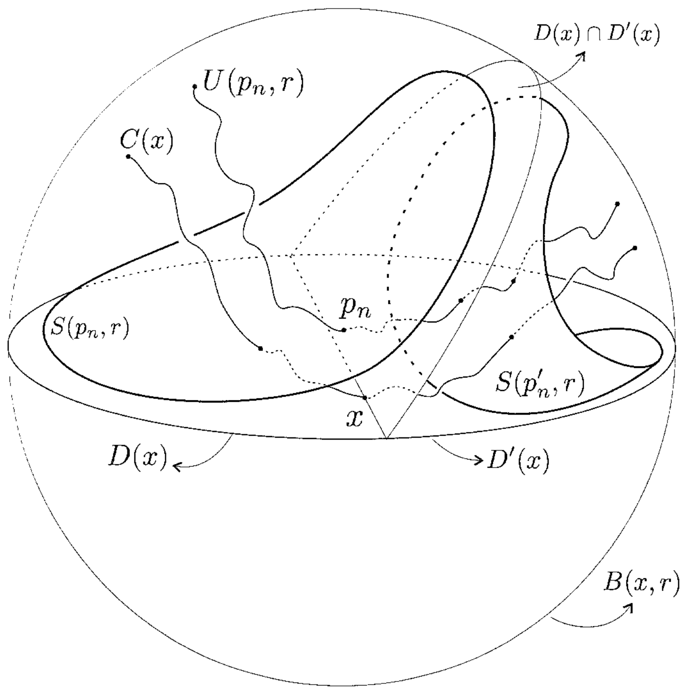


**Figure 1**

**1.** Assume that $p_n$ is of index 2. Hence $D(x)$ separates $B(x,r)$ and $B$ is contained in a single component of $B(x,r)\backslash D(x)$. For, $B$ is open-connected, and if $B \cap D(x) \neq \varnothing$ then $B \cap S(p_n, r) \neq \varnothing$ for $n$ large enough. Therefore, by our previous remark, there is an open set $N \subset B$ in which there is an $f$-lps contradicting our choice of $B$.

**2.** In at least one component of $B(x,r)\backslash D(x)$, say $X$, there is an $f$-lps. Thus $B \cap X = \varnothing$.

**3.** There is $N \in \mathbb{N}$ such that for $n \geq NU(p_n, r)$ is separated by $D(x)$. Therefore there is an $f$-lps in an open set $Y \subset B(x,r)\backslash D(x)$, such that $X \cap Y = \varnothing$ and $x \in \partial Y$. To see it, let us intersect the connected neighborhood $N$ of $U(p_n, r) \cup S(p_n, r)$ in which there is an $f$-lps with $B(x,r)\backslash(D(x) \cup X)$. Taking the union in $n \geq N$ of the sets $N$ we find $Y$. By 2.15 there is a sequence $p'_n \subset Per_H(f) \cap Y$ converging to $x$.

**4.** We use 2.13 to construct with $\{S(p'_n, r)\}$ and $\{U(p'_n, r)\}$ another pair of continua $D'(x) \subset W^s_\epsilon(x)$ and $C'(x) \subset W^u_\epsilon(x)$ in such a way that $B$ is between $D(x)$ and $D'(x)$ and $C'(x)$ contains a subcontinuum $C''(x)$ reaching $\partial B(x,r)$ and between $D(x)$ and $D'(x)$ too.

**5.** We are able to iterate the above construction finding sequences $D_n(x) \subset W^s_\epsilon(x)$ and $C_n(x) \subset W^u_\epsilon(x)$ such that $C_{n+1}(x)$ is between $D_n(x)$ and $D_{n+1}(x)$. This yields a contradiction, for it implies the existence of a common point of $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ in the boundary of $B(x,r)$ contradicting the expansive properties of $f$.

We need the following lemmas.

**Lemma 3.2.** *Let $D(x)$, $C(x)$ and $X$ be as in 2.13. Then we have:*

a) *$D(x)$ contains a 2-disk $\Delta$, $x \in \Delta$.*

b) *$C(x)$ contains an arc $\gamma$, $x \in \gamma$.*

c) *$\partial X = D(x)$.*

d) *There is a neighborhood $V$ of $x$ such that if $y \in V \cap D(x)$ and $z \in V \cap C(x) \cap X$ then $C(y)$ intersects $D(z)$ in a single point $w \in X$.*

e) *The map $h\colon (D(x) \cap V) \times (C(x) \cap V \cap X) \to B(x,r)$ defined as $h(y,z) = w$ is a homeomorphism such that $h(y,x) = y, h(x,z) = z$ and which defines an $f$-lps in $V \cap X$.*

**Proof. a)** Let us prove that $D(x)$ contains a 2-disk $\Delta$, $x \in \Delta$. Take $N_n$ a connected neighborhood of $U(p_n, r) \cup S(p_n, r)$ with an $f$-lps. For $n$

large enough, by 2.13, we may assume that $D(x)$ separates $U(p_n, r)$ and $S(p_n, r)$ separates $C(x)$. Using 2.8 we find numbers $\lambda > 0$, $\mu > 0$ such that if $y, z \in B(x, \lambda)$ then $\text{dist}(W^s_\epsilon(z) \cap \partial B(x, r), W^u_\epsilon(y) \cap \partial B(x, r)) > \mu$; take $n$ so large that $H \,\text{dist}(S(p_n, r), D(x)) < \mu$. If we have a point $y \in S(p_n, r) \cap B(x, \lambda)$ then $y \in N_n$ and therefore it is the limit of a sequence $\{q_n\} \subset Per_H(f)$. Thus, by 2.13, we have a continuum $C(y) \subset W^u_\epsilon(y)$ separated by $S(p_n, r)$ in two subcontinua such that each of them reaches the boundary of $B(x, r)$ if we assume that $\lambda$ is sufficiently small. We claim that one of the subcontinua intersects $D(x)$. Otherwise we will have points of $W^u_\epsilon(y) \cap \partial B(x, r)$ and $W^s_\epsilon(x) \cap \partial B(x, r)$ at a distance less than $\mu$ contradicting 2.8. Moreover, expansiveness implies that $C(y) \cap D(x)$ is a single point, say $w$. Let us define the map $g: S(p_n, r) \cap B(x, \lambda) \to D(x)$ sending $y$ to $C(y) \cap D(x) = w$. We have that $g$ is injective. If $y, y' \in S(p_n, r) \cap V$ have the same image $w$ then, as they are in $W^s_\epsilon(p_n)$ we have that $\text{dist}(f^m(y), f^m(y')) \leq 2\epsilon < \alpha$ for $m \geq 0$; as they are in $W^u_\epsilon(w)$ we also have that $\text{dist}(f^m(y), f^m(y')) < \alpha$ for $m \leq 0$. Therefore $\text{dist}(f^m(y), f^m(y')) < \alpha$ for $m \in \mathbb{Z}$ and therefore $y = y'$. We also have that $g$ is continuous: let $\{y_k\}$ be a sequence in $S(p_n, r) \cap B(x, \lambda)$ converging to $y \in S(p_n, r) \cap B(x, \lambda)$. The sequence $\{w_k\}$ given by $C(y_k) \cap D(x) = w_k$ has a convergent subsequence $\{w_{k_l}\}$, say $\lim_{k \to +\infty} w_{k_l} = w_\infty$. Hence $w_\infty \in D(x) \cap W^u_\epsilon(y)$ and therefore by expansiveness $w_\infty = w = C(y) \cap D(x)$. Thus $\lim_{k \to +\infty} w_k = \lim_{k \to +\infty} g(y_k) = g(y) = w$. From the existence of $g$ it follows that $D(x)$ contains a 2-disk $\Delta$. As $S(p_n, r)$ intersects $C(x)$ it is easy to see that $x \in \Delta$ for, choosing $p_n$ sufficiently close to $x$ we will have that $C(x) \cap S(p_n, r)$ is a point in $B(x, \lambda)$.

**b)** The proof of this part is similar to that of a). See [**Vi2**] §2 for details.

**c)** As there are infinitely many points of $\{p_n\}$ in $X$ and $H \lim S(p_n, r) = D(x)$ this is a simple remark.

**d)** Finally we prove that there exists a neighborhood $V$ of $x$ such that if $y \in V \cap D(x)$ and $z \in V \cap C(x) \cap X$ then $C(y)$ intersects $D(z)$ in a single point $w \in X$. From this it will follow the existence of an $f$-lps in $X$.

Let $z \in C(x) \cap X$, if $z$ is close enough to $x$, we may find points $p_n$ and

$p_{n+h}$ such that $z$ is between $S(x, p_n, r)$ and $S(x, p_{n+h}, r)$, for it holds that there is $N \in \mathbb{N}$ such that for $n \geq N$, $h \in \mathbb{N}$, $S(x, p_n, r)$ and $S(x, p_{n+h}, r)$ separate $B(x, r)$. Moreover, there is a sequence $\{q_n\}$ of thp points such that $\lim_{n\to+\infty} q_n = z$. For $z \in H \lim U(p_n, r)$ and as $p_n \in Per_H(f)$ we have that there is a neighborhood $N_n$ of $U(p_n, r)$ in which there is defined an $f$-lps. By 2.15 and 2.16 thp points are dense in $N_n$ and from this the existence of $\{q_n\}$ follows. We conclude, using 2.13, that there are continua $C(z) \subset W^u_\epsilon(z)$ and $D(z) \subset W^s_\epsilon(z)$ such that $z = C(z) \cap D(z)$, $D(z)$ separates $B(x, r)$ and $C(z)$ intersects $\partial B(x, r)$ in two connected components of $B(x, r) \backslash D(z)$. It is clear that $D(z)$ is between $S(x, p_n, r)$ and $S(x, p_{n+h}, r)$ so that there exists $N \in \mathbb{N}$ such that for $n \geq N$, $C(z)$ must intersect both $S(x, p_n, r)$ and $S(x, p_{n+h}, r)$. Otherwise, as $C(z)$ intersects $\partial B(x, r')$ we will have a sequence of points $z_n \in C(x) \cap X$ converging to $x$ and a sequence $w_n \in C(z_n) \cap \partial B(x, r')$ converging to a point $w \in D(x) \cap W^u_\epsilon(x)$ (by compactness of $M$ and taking into account that $w_n$ is between $S(x, p_n, r)$ and $S(x, p_{n+h}, r)$ which converge with $n$ to $D(x)$). But this violates the expansivity of $f$.

Let now $y$ be a point in $D(x)$. Then $y \in H \lim S(x, p_n, r)$ and as for every $n$ there is a neighborhood $N_n$ of $S(x, p_n, r)$ in which there is defined an $f$-lps we have that there is a sequence $\{q_n\} \subset X \cap Per_H(f)$ such that $\lim q_n = y$, so that there exists a compact connected set $C^+(y) \subset W^u_\epsilon(y) \cap X$ which reaches the boundary of $B(x, r)$. We will have that there is $N \in \mathbb{N}$ such that if $n \geq N$ then $C^+(y)$ intersects $S(x, p_n, r)$.

Otherwise, as we state above, the violate expansivity. From this and the fact that $D(z)$ is between $S(x, p_n, r)$ and $S(x, p_{n+h}, r)$ we will have that $D(z)$ intersects $C^+(y)$ if both $z$ and $y$ are close enough to $x$, i.e.: they belong to a neighborhood $V$ of $x$. Expansiveness implies that $C^+(y) \cap D(z) = w$ a single point.

**e)** Define $h\colon (V \cap D(x)) \times (V \cap C(x) \cap X) \to X$ by $h(y, z) = w$ where as above $w = C^+(y) \cap D(z)$. As in **[Vi1]** §5, we may prove that $h$ is a continuous injective map. In view that $D(x) \cap V$ contains a 2-disk and $C(x) \cap V$ contains an arc, by the Theorem of Invariance of Domain, $h$ is a homeomorphism. By the definition of $h$ it is clear that it extends as the inclusion map in $D(x) \cap V$ and in $C(x) \cap X \cap V$. As in [**Vi2**], §§3 and 4, we may see that we have an $f$-lps defined on $X \cap V$. This finishes the proof. □

**Corollary 3.3.** *If $V$ is homeomorphic to $B^3$ then the connected component of $D(x) \cap V$ containing $x$ is a 2-disk and the connected component of $C(x) \cap X \cap V$ is an arc.*

**Proof.** It follows from the existence of the $f$-lps in $X \cap V$ and the fact that the map $h$ in lemma 3.2 extends as the inclusion to $clos(X)$. □

**Lemma 3.4.** *Let $Y$ be the other region of $B(x,r)\backslash D(x)$, different from $X$, in which there are points of $C(x)$. Then, for every neighborhood $V$ of $x$, there are points $p' \in V \cap Y \cap Per_H(f)$. Hence there is a sequence $\{p'_n\} \subset Y \cap Per_H(f)$ with $\lim_{n\to+\infty} p'_n = x$.*

**Proof.** Let $\{p_n\} \subset Per_H(f)$ be as in 2.13 such that $H \lim U(p_n, r) = C(x)$. We have that there is $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$U(p_n, r)\backslash clos(X) = U(p_n, r) \cap Y \neq \varnothing.$$

Moreover, given $V$ an open neighborhood of $x$, there will be points of $U(p_n, r)$ in $Y \cap N$ if $N$ is large enough. There is a neighborhood $N_n$ of $U(p_n, r)$ in which there is defined an $f$-lps. But $V \cap Y \cap N_n$ is open so, by Proposition 2.2, there exists $p' \in V \cap Y \cap Per_H(f)$. As $V$ is an arbitrary neighborhood of $x$ we have that there is a sequence $\{p'_n\} \subset Y \cap Per_H(f)$ with $\lim_{n\to+\infty} p'_n = x$. □

We will call $Z$ to $B(x,r)\backslash clos(X)$, so $Y \subset \mathbb{Z}$.

Let $\{p'_n\} \subset Y$ be as in 3.4. Using 2.13 and taking subsequences if it were necessary we obtain continua $D'(x) = H \lim_{n\to+\infty} S(p'_n, r)$ and $C'(x) = H \lim_{n\to+\infty} U(p'_n, r)$. Let $X'$ be the connected component of $B(x,r)\backslash D'(x)$ in which there is defined an $f$-lps and $Z' = B(x,r)\backslash clos(X')$.

**Lemma 3.5.** *We have that $H\,\mathrm{dist}(D(x), D'(x)) > 0$. Moreover, $B$ is between $D(x)$ and $D'(x)$ and for every neighborhood $V$ of $x$, nor $D'(x) \cap V \subset D(x) \cap V$ neither $D(x) \cap V \subset D'(x) \cap V$.*

**Proof.** We may assume that $B \subset B(x,r)$. As $S(x,p'_n,r)$ and $B$ are both included in $Z$ for all $h \in \mathbb{N}S(x,p_h,r) \subset X$, there is $N \in \mathbb{N}$ such that for all $n \geq N$ they are in the same region of $B(x,r)\backslash S(x,p_h,r)$, for both $B$ and $S(p'_n,r)$ may be joined by an arc with $D(x)$ without intersections with $S(x,p_h,r)$. Similarly there is $H \in \mathbb{N}$ such that $S(x,p_h,r)$ and $B$ are both in $Z'$ and in the same region on $B(x,r')\backslash S(x,p'_n,r)$ if $h \geq H$. Thus $B$ is between $S(x,p'_n,r)$ and $S(x,p_h,r)$ for all $n$ and $h$ sufficiently large and from this it follows that $B$ is between $D(x)$ and $D'(x)$. Suppose that there is a neighborhood $V$ of $x$, $V \subset B(x,r)$, such that $D(x) \cap V \subset D'(x) \cap V$. Hence for all $\sigma > 0$ there is $N \in \mathbb{N}$ such that for all $n$, $h \geq N$ it holds that $S(x,p_h,r) \cap V \subset [S(x,p'_n,r)]_\sigma$ the $\sigma$-parallel body of $S(x,p'_n,r)$. Therefore, as $B$ is between $S(x,p'_n,r)$ and $S(x,p_h,r)$, $B \cap V \subset [S(x,p',r)]_\sigma$ for all $\sigma > 0$ which contradicts the fact that $B$ is open and $x \in \partial B$. □

**Remark 3.6. 1.** We have proved a bit more than 3.5 says: for every neighborhood $V$ of $x$, $V \cap B$ is between $D(x) \cap V$ and $D'(x) \cap V$.

**2.** By 2.15 and 2.16 the thp-points are dense in $X \cup X'$ and we may consider a sequence $\{q'_n\}$ converging to the boundary $\partial(X \cup X') = D(x) \cup D'(x)$. We will show that we may choose $U(q'_n,r)$ such that it intersects $D(x) \cup D'(x)$ and enters into $B(x,r)\backslash(X \cup X')$ in the region $\mathcal{B}_0$ of $B(x,r)\backslash(D(x) \cup D'(x))$ containing $B$.

**Lemma 3.7.** *There is a sequence* $\{q_n\} \subset Per_H(f) \cap \mathcal{B}_0$ *such that*

$$\lim_{n\to\infty} q_n = x.$$

**Proof.** By the remark above, for every neighborhood $V$ of $x$ we have $V \cap B \neq \varnothing$. On the other hand, if $clos(X \cup X') \cap V = V \cap B(x,r)$ then $V \cap B = \varnothing$ contradicting that $x \in \partial B$. Let $z \in V$, $z \neq x$, be a point in the boundary of $\mathcal{B}_0$. Hence $z$ is in $\partial(X \cup X') = D(x) \cup D'(x)$ too and there is a sequence $\{q'_n\} \subset Per_H(f) \cap (X \cup X')$ such that $lim_{n\to+\infty} q'_n = z$. Assume that $z \in D(x)$ (if it belongs to $D'(x)$ we simply change $X$ by $X'$ in the arguments following below). Hence we may take $\{q'_n\} \subset X$ and from this it is not difficult to prove that $U(q'_n,r)$ intersects $D(x)$ near $z$ as $n \to +\infty$.

In fact, if this is not true, we will have that either $U(q'_n, r)$ is between $D(x)$ and $S(q'_n, r)$ or $U(q'_n, r)$ intersects $D(x)$ at a distance bounded away from zero for infinitely many values of $n$. In the first case we will contradict 2.8, and in the second one we will have two points of intersections of $W^u_\epsilon(z)$ with $D(x)$, contradicting expansivity. Thus $U(q'_n, r)$ enters in $\mathcal{B}_0$ and this implies that there is a point $w_n \in \mathcal{B}_0 \cap U(q'_n, r)$, $\lim w_n = z$. As there is an open set $N_n \supset U(q'_n, r)$ in which there is an $f$-lps we conclude from 2.15 and 2.16 that there is a point $q_n \in Per_H(f) \cap \mathcal{B}_0$ near $w_n$. Hence, as $V$ is an arbitrary neighborhood of $x$, we have that there is a sequence $\{q_n\} \subset Per_H(f) \cap \mathcal{B}_0$ such that $\lim_{n\to+\infty} q_n = x$. □

**Lemma 3.8.** *There are continua $C''(x) \subset W^u_\epsilon(x)$ and $D''(x) \subset W^s_\epsilon(x)$ such that $C''(x)$ contains a subcontinuum $C''_+$ included in $\mathcal{B}_0$, $x \in C''_+$, reaching $\partial B(x,r)$ and $D''(x)$ separates $B(x,r)$ and is included in $clos(\mathcal{B}_0)$, $x \in D''(x)$. If $X''$ is the region of $B(x,r)\backslash D''(x)$ in which there are infinitely many points of the sequence $\{q_n\} \subset Per_H(f) \cap \mathcal{B}_0$ of 3.7 then there is a neighborhood $V_2$ of $x$ in which there is defined an $f$-lps. For all neighborhood $V$ of $x$ neither*

$$D''(x) \cap V \subset D(x) \cap V (D''(x) \cap V \subset D'(x) \cap V)$$

*nor*

$$D(x) \cap V \subset D''(x) \cap V \quad (\textit{resp.} \quad D'(x) \cap V \subset D''(x) \cap V).$$

**Proof.** Taking a convergent subsequence from

$$\{U(q_n, r)\}, \{q_n\} \subset Per_H(f) \cap \mathcal{B}_0,$$

we find the desired compact connected sets $C''(x)$ and $D''(x)$. As in 2.13 we prove the existence of $V_2$. Observe that $B$ is between $D(x)$ and $D''(x)$ and also between $D'(x)$ and $D''(x)$ and repeat the arguments of 3.5 to prove that for every neighborhood $V$ of $xD''(x) \cap V$ is not included in $D(x) \cap V$. □

**Remark 3.9.** **1.** There is only one region, say $\mathcal{B}_2$, of $B(x,r)\backslash(D(x) \cup D'(x) \cup D''(x))$ that contains $B$. Otherwise for a point $p \in Per_H(f)$

of one of the sequences used to construct $D(x)$, $D'(x)$ or $D''(x)$ we will have that $S(p,r)$ intersects $B$.

**2.** $C''$ is included in $\mathcal{B}_0$ and therefore it is between $D(x)$ and $D'(x)$. If we call $C_+$ to $C(x)\cap X$ and $C'_+$ to $C'(x)\cap X'$, then $C_+$ is between $D'(x)$ and $D''(x)$ and $C'_+$ is between $D(x)$ and $D''(x)$. Thus between every pair of separating continua, like $D(x)$, included in $W^s_\epsilon(x)$ we have a continuum, like $C(x)$, included in $W^u_\epsilon(x)$. Observe that, in spite of the size of the neighborhood $V_2$ in which there is an $f$-lps, $D(x)$, $D'(x)$ and $D''(x)$ separate $B(x,r)$, and $C_+(x)$, $C'_+(x)$ and $C''_+(x)$ reach $\partial B(x,r)$.

**3.** We may iterate the same procedures used in the previous lemmas to construct a sequence of continua $D_n(x)\subset W^s_\epsilon(x)\cap\mathcal{B}_2$ separating $B(x,r)$ and $C_n(x)\subset W^u_\epsilon(x)$ such that it contains a subcontinuum $C_n^+$ between $D_n(x)$ and $D_{n+1}(x)$, which reaches $\partial B(x,r)$.

**Theorem 3.10 (Theorem A).** *Let $f\colon M\to M$ be an expansive homeomorphism (see definitions below in this section). Assume that the non wandering set of $f$, $\Omega(f)$, is $M$ and that there exists an open set $V\subset M$, $V\neq\varnothing$, such that there is an $f$-lps in $V$. Then $f$ is conjugate to a linear Anosov isomorphism and $M\simeq T^3$.*

**Proof.** If $B$ exists, by the previous lemmas, we may construct sequences of continua

$$D_1(x)=D(x),D_2(x)=D'(x),D_3(x)=D''(x),\dots,D_n(x),\dots,$$

all $D_n(x)\subset W^s_\epsilon(x)$ separating $B(x,r)$, and

$$C_1(x)=C''(x),C_2(x)=C(x),C_3(x),\dots,C_n(x),\dots,$$

all $C_n(x)\subset W^u_\epsilon(x)$, such that for all $n\in\mathbb{N}C_n(x)$ contains a subcontinuum $C_n^+$ which is between $D_n(x)$ and $D_{n+1}(x)$ and reaches $\partial B(x,r)$. Thus we will have a common limit point $w\in\partial B(x,r)$ of $\{D_n(x)\}$ and $\{C_n(x)\}$, which implies that $w\in W^u_\epsilon(x)\cap W^s_\epsilon(x)$, $w\neq x$, contradicting that $f$ is expansive. □

## 4. Proof of Theorem B

We will assume that $p$ is a hyperbolic fixed point of index 2 and that $x\in$

$W^s(p) \cap W^u(p)$, $x \neq p$. The general case in which $p$ is only periodic and $x \in W^s(\mathcal{O}(p)) \cap W^u(\mathcal{O}(p))$, $x \neq p$ may be obtained from this particular one.

We may take linearizing coordinates and assume that, in a certain neighborhood $B(p, r)$ of $p$ in which Hartman-Groβman Theorem is valid, $S(p, r)$ is a disc centred in $O$ in the $Oxy$ plane and $U(p, r)$ is an arc, centred in $O$ too, in the $Oz$ axis.

**Definition 4.1.** We say that a set $D$ locally separates a set $A$ at a point $y$, $y \in A \cap D$, if there is a neighborhood $V$ of $y$ such that for every neighborhood $W$ of $y$, $W \subset V$, $W \cap A$ has points in different components of $W \backslash D$.

Let $S_n(r)$ be the connected component of $W^s_\epsilon(f^{-n}(x)) \cap B(f^{-n}(x), r)$ that contains $f^{-n}(x)$.

**Lemma 4.1.** *Let $p$ be a hyperbolic fixed point and $x \in W^s(p) \cap W^u(p)$, $x \neq p$. Then there are $r_0 > 0$, and $N \in \mathbb{N}$ such that for $n \geq N$, $0 < r \leq r_0$, $S_n(r)$ separates $B(f^{-n}(x), r)$. (See Figure 2 for illustration).*

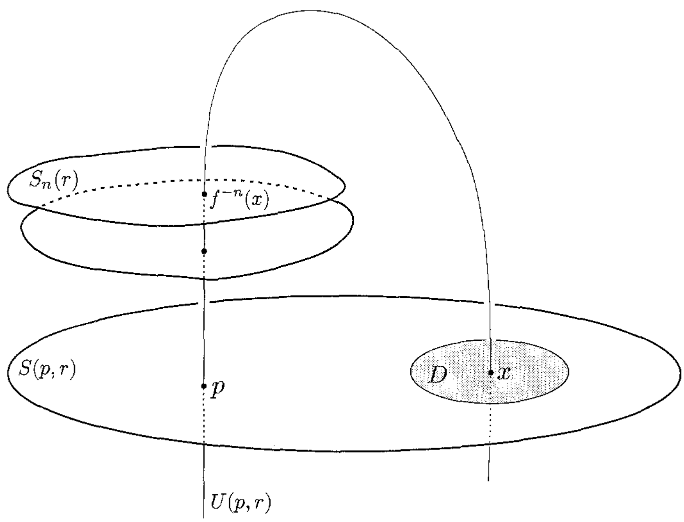


**Figure 2**

**Proof.** As $W^j_\epsilon(x) \subset W^j(p)$, $j = s, u$, we have that there is a 2-disk $D = g(D^2)$, where $D^2$ is the standard 2-disk and $g$ is an embedding, such that $D = \int(W^s_\epsilon(x))$, $x \in D$, and $W^u_\epsilon(x)$ is an arc if $\epsilon$ is less than $\alpha$.

As every topological disk locally separates $\mathbb{R}^3$ (see [**Bi**]), we have that $D$ locally separates a neighborhood $V$ of $x$. Let $\gamma \subset D$ be a continuous arc joining $x$ with $\partial D$, i.e.: $\gamma: [0,1] \to D$, $\gamma(0) = x$, and $\gamma(1) \in \partial D$. We will prove that there is $N \in \mathbb{N}$ such that for all $n \geq N$ and all arc $\gamma$ joining $x$ with $\partial D$, $f^{-n}(\gamma)$ intersects $\partial B(f^{-n}(x), r)$ if $0 < r \leq r_0$, $r_0$ to be chosen, this will enable us to prove the thesis. We use arguments similar to those used in [**Vi1**] to prove 2.11. We choose Lyapunov functions $U$, $U'$ and $U''$ for the suspension flow $\phi$ of $f$, as in remark 2.3 and positive numbers $k$, $r$, such that for $x$, $y \in M \simeq \pi(M \times 0)\,\mathrm{dist}(x,y) < r_0 \Rightarrow U(x,y) < k$ and $U(x,y) < k \Rightarrow \mathrm{dist}(x,y) < \epsilon < \frac{\alpha}{2}$. Define a Lyapunov tube for $x$ as $K(x) = \bigcup_{t \in \mathbb{R}} K_t(x)$ where $K_t(x) = \{y \in \tilde{M}/y \in M_t \quad \text{and} \quad U(\phi_t(x), y) \leq k\}$.

Let also $\mathrm{int}(K_t(x)) = \{y \in M_t/U(\phi(x,t),y) < k\}$ and $\partial K_t(x) = \{y \in M_t/U(\phi(x,t),y) = k\}$. The disk $D$ is included in $W^s_\epsilon(x)$ hence if $y \in D$ then we have that $\mathrm{dist}(f^n(x), f^n(y)) < \alpha$ for all $n \geq 0$. Thus for every $y \in D$, $y \neq x$, we must have that there is $n(y) > 0$ such that $f^{-n(y)}(y) = \phi(y, -n(y)) \in /K_{-n(y)}(x)$, otherwise expansivity is violated.

By compactness of $\partial D$, there is $N > 0$ such that if $y \in \partial D$ then $n(y) \neq N$. We claim that if $y \in \partial D$ and $\gamma$ joins $x$ with $y$ then for all $T \geq N$ there is a point $s^* = s^*(T) \in (0,1]$ such that $\phi(\gamma(s^*), -T) \in / \int(K_{-T}(x))$. If this were not true we would have that $\gamma = \phi(\gamma, 0) \subset \int(K_0(x))$ and $\phi(\gamma, -T)\int(K_{-T}(x))$ but $\phi(y, -n(y)) \in /K_{-n(y)}(x)$ so there will be a point $\gamma(s_0)$ such that $\phi(\gamma(s_0), -t) \in K^{-t}(x)$, $t \in [0,T]$ and there is $t_0 \in (0,T)$ such that $\phi(\gamma(s_0), -t_0) \in \partial K_{-t_0}(x)$ which contradicts the fact that $U''(\phi(x,-t_0), \phi(\gamma(s_0), -t_0))$ is positive. It is clear, by continuity of $\phi$ and $U$, that for all $t \geq N\phi(\gamma, -t)$ intersects $\partial B(\phi(x,-t), r)$ if $0 < r \leq r_0$.

Using the fact just proved we are able to prove that $\phi_{-t}(g(D^2))$ separates $B(\phi_{-t}(x), r_0)$, moreover, $S_t$ the connected component of $\phi_{-t} \circ g(D^2) \cap B(\phi_{-t}(x), r_0)$ containing $\phi_{-t}(x)$ does it. In order to prove this observe that:

1. $B(\phi_{-t}(x), r_0)$ is open in $M$ and $B(\phi_{-t}(p), r_0) \simeq B^3 = \{(x,y,z) \in$

$\mathbb{R}^3/x^2+y^2+z^2<1\}$ the standard 3-cell by our choice of $r_0$.

2. By 1., the definition of $S_t$ and the fact that $\phi_{-t}\circ g$ is an embedding, $g^{-1}\circ\phi_t(S_t)$ is open and connected in $D^2$, which is locally arcwise connected. Thus $g^{-1}\circ\phi_t(S_t)$ is arcwise connected.
3. $g^{-1}\circ\phi_t(S_t)\cap\partial D^2=\varnothing$, otherwise there would exist an arc $\gamma$ joining $(0,0)$ with $\partial D^2$ such that $\phi_{-t}\circ g(\gamma)\subset B(\phi_{-t}(x),r_0)\subset K_{-t}(x)$. Therefore $g^{-1}\circ\phi_t(S_t)$ is open in $\mathbb{R}^2$.
4. By definition $S_t$ is closed in $B(\phi_{-t}(x),r_0)$ and, by 3., it is the image of an open 2-manifold by the embedding $\phi-t\circ g$.
5. Thus we have reduced the proof to the following fact:

If $F$ is closed in $B=B^3$ and is the image of an open surface included in $\mathbb{R}^2$ by a topological embedding, then it separates $B$. To prove the last statement we will use rational homology.

Let $X$ be an orientable manifold and $U$ an orientable submanifold $U\subset X$. Denote the absolute $n^{th}$ homology group by $H_n(X)$, the relative $n^{th}$ homology group with respect to $U$ by $H_n(X,U)$ and the $n^{th}$ cohomology group with compact support by $H_c^n(X)$. Recall that if $\dim(X)=s$ then by Alexander-Pontrjaguin's Duality Theorem we have that $H_n(X,U)=H_c^{s-n}(X\backslash U)$ (see [**Sp**] Chs. 4, 5 and 6).

Now let $X=B$ and $U=B\backslash F$. As $F$ is closed in $B$, $U$ inherits from $B$ the structure of an orientable submanifold. By the Exact Homology Sequence (see [**Sp**] Ch. 4) we have that:

$$\to H_1(B)\to H_1(B,U)\to H_0(U)\to H_0(B)\to H_0(B,U)\to(0)$$

As $U\neq\varnothing$ and $B$ is arcwise connected we have $H_0(B,U)=(0)$ and also as $B$ is a 3-cell $H_1(B)=(0)$ thus we have the short exact sequence

$$(0)\to H_1(B,U)\to H_0(U)\to H_0(B)\to(0).$$

Again as $B$ is a 3-cell $H_0(B)\simeq\mathbb{R}$ and, by Duality, $H_1(B,U)=H_c^2(F)=H_0(F,\varnothing)=H_0(F)\simeq\mathbb{R}$, where we have used that $F$ is orientable being the image of an open set of $\mathbb{R}^2$ under an embedding. Therefore we have that $H_0(U)\simeq\mathbb{R}^2$ and this implies that $U$ has two connected components.

To finish the proof we observe that $S_t$ is contained in $W_\epsilon^s(\phi_{-t}(x))$. Moreover $\phi_\tau(S_t)\subset K_{\tau-t}(x)$ for all $\tau\geq 0$. If it were not true, as for

$\tau = t\phi_t(S_t) \subset g(D^2) \subset K_0(x)$, then we could find a $\phi$-trajectory through a point in $S_t$ tangent to the tube $K(x)$ contradicting that $U'' > 0$. $\square$

Let $r_0 > 0$ and $N \in \mathbb{N}$ be such that for $0 < r \leq r_0$, $S_n(r)$ separates $B(f^{-n}(x), r)$ for all $n \geq N$.

**Lemma 4.2.** *Let $p$ be a hyperbolic fixed point and $x \in W^s(p) \cap W^u(p)$, $x \neq p$. Then given $r > 0$, $0 < r \leq r_0$, there is $r_1$, $0 < r_1 \leq r$ and $N \in \mathbb{N}$ such that for all $n \geq N$, and $0 < r' \leq r_1$, every closed curve $\gamma$ contained in $S_n(r) \cap B(f^{-n}(x), r')$ is homotopically trivial within $S_n(r)$.*

**Proof.** Observe first that fixing $n$ the thesis is trivial, for $f^{-n}(x)$ is an interior point of $S_n(r)$, which is locally $\mathbb{R}^2$. Moreover, it is clear that if the thesis holds for $r$ then the same is true for $r''$, $r \leq r'' \leq r_0$; this follows from the fact that $S_n(r) \subset S_n(r'')$. By Sard's Theorem the set of $r$ such that $W^s_\epsilon(f^{-n}(x)) \cap B(f^{-n}(x), r)$ is a non singular set (i.e.: a finite union of smooth curves) is a residual one. Suppose that the thesis is false and observe that, by our previous remark, we may assume that $r$ is fixed and the same for all $n$, and that $S_n(r) \cap B(f^{-n}(x), r)$ is non singular.

Therefore we will have that for all $r_1$ there exist $r'$, $0 < r' \leq r_1$, and $S_{n_h}(r)$ such that there is a sequence of non singular closed curves $\gamma_h$ in $S_{n_h}(r) \cap B(f^{-n_h}(x), r')$, with $\lim_{h\to\infty} n_h = \infty$, such that $H\lim_{h\to\infty} \gamma_h = p$, and a sequence of points $\{z_h\} \subset \partial B(f^{n_h}(x), r_0)$ such that $z_h$ belongs to the 2-cylinder $C_h \subset W^s_\epsilon(f^{-n_h}(x))$ such that $\partial C_h = \gamma_h \cup \beta_h$, $z_h \in \beta_h$, where $\beta_h$ is a non singular closed curve in the intersection of $S_{n_h}(r)$ with $\partial B(f^{-n_h}(x), r)$. As $\lim_{h\to\infty} f^{-n_h}(x) = p$ we have that $W^s_\epsilon(f^{-n_h}(x)) \subset [W^s_\epsilon(p)]_\sigma$ so that $C_h$ is also included in $[W^s_\epsilon(p)]_\sigma$ with $\sigma > 0$ small if $h$ is large enough. Let us close $C_h$ with disks $G_h$ and $G'_h$, $\partial G_h = \gamma_h$, $\partial G'_h = \beta_h$, such that $\text{dist}(G_h, G'_h) > r/2$. Take a point $w_h$ in the interior of $C_h \cup G_h \cup G'_h$ and another disk $D'_h$, $w_h \in D'_h$, at a distance $> r/4$ from $G_h \cup G'_h$ such that $\partial D'_h = \gamma'_h \subset C_h$ is a non-contractible curve in $C_h$. By 2.6, for every point $x \in M$ there is a continuum $C(z) \subset W^u_\epsilon(z)$ which intersects $\partial B(z, r)$. If $z \in D'_h$ then, by 2.8, choosing $\sigma$ small enough, $C(z) \cap G_h \cup G'_h = \varnothing$ and therefore $C(z)$ intersects $C_h$ in a single point (if we have 2 points of intersection then expansivity is violated).

Thus the map $H\colon D'_h \to C_h$ such that $H(z) = w$ where $\{w\} = C(z) \cap C_h$ is well defined and continuous: let $\{z_n\} \subset D'_h$, $\lim z_h = z \in D'_h$. Let $\{w_n\} = C(z_n) \cap C_h$. As $\{w_h\}$ is a bounded sequence it has a convergent subsequence so let $\{w_{n_k}\}$ converge to $w$. Clearly $w \in C_h \cap W^u_\epsilon(z)$ so by the expansivity of $f$ it holds that $\{w\} = C(z) \cap C_h$. Thus $\lim H(z_n) = H(z)$ and $h$ is continuous. If $z \in \gamma'_h$ then $H(z) = z$, hence $\gamma'_h \subset H(D'_h)$.

But, by continuity of $H$ and as $D'_h$ is a 2-disk, we have that $H(D'_h)$ is contractible in $C_h$ contradicting the fact that $\gamma'_h$ does not bound a 2-disk in the cylinder $C_h$. Therefore there must exist $r_1$ with the desired properties. □

**Theorem 4.3 (Theorem B).** *The stable manifold of $p$, $W^s_\epsilon(p)$, is topologically transverse to its unstable one, $W^u_\epsilon(p)$, at the point $x$.*

**Proof.** First we prove that $W^s_\epsilon(x)$ locally separates $W^u_\epsilon(x)$. If it is not true then there is a neighborhood $\mathcal{W}$ of $x$ such that $\mathcal{W} \cap W^u_\epsilon(x)$ is contained in a single region of $\mathcal{W} \backslash W^s_\epsilon(x)$. This implies that $W^u_\epsilon(x)$ is tangent to $W^s_\epsilon(x)$ at $x$. As $x$ is a homoclinic point we have that $\lim_{n\to\pm\infty} f^n(x) = p$ and $W^u(p)$ will be tangent to $W^s(p)$ at $f^n(x)$, $n \in \mathbb{Z}$.

Let us now give a brief idea of the proof in the two-dimensional case (see Figure 3). In this case, $S_n(r)$ will be an arc tangent to $W^u(p)$ at $f^{-n}(x)$ for $n$ sufficiently large. Moreover, $S_n(r)$ will have two prongs, say $\lambda$ and $\mu$, separated by $f^{-n}(x)$ such that both of them are close to $W^s_\epsilon(p)$. If we take an arc $\beta$ in the thin region of $B(f^{-n}(x), r) \backslash S_n(r)$ joining both prongs and at middle distance between $f^{-n}(x)$ and the boundary of $B(f^{-n}(x), r)$, the existence of a continuum $C(z) \subset W^u_\epsilon(z)$ for all $z \in \beta$, such that $C(z)$ intersects $\partial B(f^{-n}(x), r)$ will lead us to a contradiction.

For, if $\sigma$ is small enough, 2.8 prohibits $C(z)$ to stay between the two prongs near $\partial B(f^{-n}(x), r)$ and near $f^{-n}(x)$. Therefore $C(z)$ must intersect either $\lambda$ and $\mu$ and, by the expansive properties of $f$, only one of them.

We classify the points of $\beta$ in two classes:

$$A = \{z \in \beta / C(z) \quad \text{intersects} \quad \lambda\}$$

and

$$B = \{z \in \beta / C(z) \quad \text{intersects} \quad \mu\}.$$

We have that $A \cup B = \beta$, as $\beta$ joins $\lambda$ with $\mu$, $A$ and $B$ are non void, and both $A$ and $B$ are closed sets. Therefore $\beta$ is not connected which is absurd. The following argument is an adaptation of these ideas to the

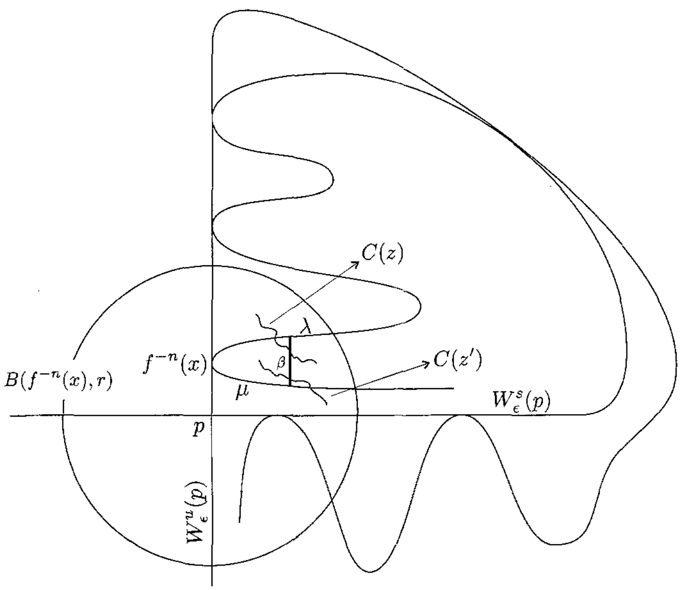


**Figure 3**

three dimensional case.

Let $N_1 \in \mathbb{N}$ be such that for all $n \geq N_1$, $S_n(r) \subset W^s_\epsilon(f^{-n}(x))$ separates $B(f^{-n}(x), r)$ with $0 < r \leq r_0$. Given $\sigma > 0$ let $N_2 \in \mathbb{N}$ be such that for all $n \geq N_2 S_n(r) \subset [W^s_\epsilon(p)]_\sigma$. Hence one of the components of $B(f^{-n}(x), r) \backslash S_n(r)$ is contained in $[W^s_\epsilon(p)]_\sigma$ too. Choose $N = \max\{N_1, N_2\}$ and let $r' \leq r_1$, $r_1$ as in 4.2. Choose a Jordan curve $\gamma$, $\gamma \subset S_n(r) \cap B(f^{-n}(x), r')$, such that $\text{dist}(\gamma, f^{-n}(x)) > r'/4$ and $\text{dist}(\gamma, \partial B(f^{-n}(x), r)) > r'/4$, $n \geq N$. By 4.2 we may choose $\gamma$ such that it bounds a 2-disk $D \subset S_n(r)$ containing $f^{-n}(x)$ in its interior. Let $D'$ be another disk such that $\partial D' = \gamma$ and such that $f(D')$ is contained in the connected component of $B(f^{-n}(x), r) \backslash S_n(r)$ with no points of $W^u_\epsilon(f^{-n}(x))$. Observe that we can take $D'$ satisfying that $\text{dist}(D', \partial B(f^{-n}(x), r)) > r'/8$ and $\text{dist}(f^{-n}(x), D') > r'/8$ too. Using 2.8 we may prove that if $\sigma$ is small enough this component is that one contained in $[W^s_\epsilon(p)]_\sigma$, otherwise we will have points of $W^u_\epsilon(f^{-n}(x))$ and $W^s_\epsilon(f^{-n}(x))$ at a distance less than $2\sigma$ in the boundary of $B(f^{-n}(x), r)$ which contradicts 2.8. Also for the same reason we have that for $\sigma > 0$ sufficiently small we will have that for all $z \in D'$ there is a continuum $C(z) \subset W^u_\epsilon(z)$ which intersects $D$ in a single point $w$ at a distance less than $r'/16$ from $D'$, hence at a distance greater than $r'/16$ from both $\partial B(f^{-n}(x), r)$ and $f^{-n}(x)$. Let $h: D' \to S_n(r)$ be such that $h(z) = w$.

As in 4.2, we have that $h$ is a continuous map and $h(\gamma) = \gamma$. Observe that $\gamma$ is not contractible in $S_n(r) \backslash B(f^{-n}(x), r'/16)$. But, by continuity of $h$, we have that $h(D') \subset S_n(r)$ is homotopically trivial which contradicts the fact that $\gamma$ is not contractible in $S_n(r) \backslash B(f^{-n}(x), r'/16)$. This finishes the proof that $W^s_\epsilon(x)$ locally separates $W^u_\epsilon(x)$.

It seems clear that from this fact, taking into account that $W^s_\epsilon(x) \cap W^u_\epsilon(x) = x$, we may derive that $W^s(p)$ is topologically transverse to $W^u(p)$ at $x$. On the other hand I cannot find bibliography for this, so let me sketch the proof here. We may assume that $x$ is the origin $O$ in $\mathbb{R}^3$ and $W^s_\epsilon(x)$ is the standard 2-disk $D$ in the $Oxy$ plane. Moreover, $W^u_\epsilon(x)$ is locally a smooth arc $\gamma$, which we parameterize with the arc length $s$, such that $\gamma(0) = O$. We may assume that for $s > 0 \gamma(s) \in H^+ = \{(x, y, z) \in \mathbb{R}^3 / z > 0\}$ and for $s < 0 \gamma(s) \in H^- = \{(x, y, z) \in \mathbb{R}^3 / z < 0\}$ (expansiveness implies that $\gamma \cap D$ is a single point and $D$ separates $\gamma$ as we have shown).

On one hand, if we have that $\gamma'_z(0) \neq 0$, where we have written $\gamma'(s) = (\gamma'_x(s), \gamma'_y(x), \gamma'_z(s))$ for the derivative of $\gamma(s)$ with respect to $s$, then the result follows from the Implicit Function Theorem. On the other hand, if $\gamma'_z(0) = 0$ we may assume, rotating the axes if it were necessary, that $(\gamma'_x(0), \gamma'_y(0), \gamma'_z(0)) = (1, 0, 0)$. Consider the plane $A(s)$ normal to $\gamma'(s)$ at the point $\gamma(s)$ and the arc of equation $\gamma_1(s) = (\gamma_x(s), \gamma_y(s), s\gamma_z(s))$, $s \in (0, \delta)$, $\delta \in (0, 1)$ to be chosen later, and let us define a positive function $\epsilon(s)$ such that $\epsilon^2(s) = (1 - s)^2 (\gamma(s))^2$, $s \in (0, \delta)$, and a compact set $V_1 = \{(x,y,z) \in \mathbb{R}^3/(x-\gamma_x(s))^2 + (y-\gamma_y(s))^2 + (z-\gamma_z(s))^2 \leq \epsilon(s), (x,y,z) \in A(s)\}$. Hence $\gamma_1(s) = \gamma(s) + (0,0,-\epsilon(s))$ is included in $V_1$ and lies in its boundary as well as

$$\gamma_2(s) = \gamma(s) + (0,0,\epsilon(s)) = (\gamma_x(s), \gamma_y(s), (2-s)\gamma_z(s)).$$

Moreover it is not difficult to see that the boundary of $V_1$ may be thought of as composed by arcs of the form $\Gamma(l,m,n)(s) = \gamma(s) + \epsilon(s)(l,m,n)$ where $l^2+m^2+n^2 = 1$, $l,m,n \in \mathbb{R}$, $s \in [0,\delta]$. We enlarge $V_1$ joining to it the solid half-sphere

$$H = \{p \in \mathbb{R}^3/\|p-\gamma(\delta)\| \leq \epsilon(\delta), < p-\gamma(\delta), \gamma'(\delta) > \geq 0\}.$$

Let us call $V_2$ to this new set.

If we have a point in $V \cap Oxy$ then we have that

$$(x-\gamma_x(s))^2 + (y-\gamma_y(s))^2 \leq \epsilon^2(s) - (\gamma_z(s))^2 = ((1-s)^2-1)(\gamma_z(s))^2 < 0$$

if $s \in (0,2)$. Thus choosing $\delta < 1$ we have that $V_2 \cap Oxy = O$. Moreover, for a fixed $s \in [0,\delta]$ we have that the point of maximum height lies in $\gamma_2(s)$ and that of minimum height lies in $\gamma_1(s)$, refine our choose of $\delta$ in order to have $\gamma_2(\delta)$ as the point of maximum height of $V_1$ and in such a way that the angle between the tangent versor $\gamma'(s)$ and $\gamma'(0) = (1,0,0)$ remains less than $\pi/8$ for all $s \in [0,\delta]$ and such that $\mathcal{L}'$, the straight line segment joining $O$ with $\gamma(\delta)$, also forms an angle less than $\pi/8$ with $(1,0,0)$. Let $\Pi(s)$ be the plane passing through $\gamma(s)$ and orthogonal to $\bar{n}$, the versor of the direction of $\mathcal{L}'$ such that $< \bar{n}, \bar{i} >> 0$ (here $\bar{i}$ is the verosr $(1,0,0)$), and let us choose another straight line segment $\mathcal{L} = [O, p_0]$, passing through $O$ and a point $p_0$ such that $p_0 = \gamma(\delta) + \lambda_0(0,0,1)$ where $\lambda_0 > 0$ has been chosen in such a way that $\mathcal{L}$ forms an angle less than $\pi/3$ with $(1,0,0)$ and such that it intersects $V_2$ only at $O$. This is possible due to our previous choice of $\delta$. Finally, let us enlarge $V_2$ by tracing the straight line segments joining points of $V_2$ to those of $\mathcal{L}$, we call $V$ to this new set. We construct a homeomorphism which is the identity in the boundary of $V$ and which "pushes" $\gamma(s)$ against $\mathcal{L}'$. We extend it to the exterior of $V$ as the identity map. It is clear that this homeomorphism is the identity in the 2-disk $D$. We repeat this construction for $s < 0$.

Thus we have reduced the last case to that in which $\gamma'_z \neq 0$ and hence we may apply the Implicit Function Theorem again. □

## 5. Proof of Theorem C

**1.** From Theorem B we may assume for the rest of this paper that for every hyperbolic periodic point $p$, $W^u(\mathcal{O}(p))$ is topologically transverse to $W^s(\mathcal{O}(p))$ at every homoclinic point $x$, $x \in W^s(\mathcal{O}(p)) \cap W^u(\mathcal{O}(p))$. From the results of 4 we have that there are numbers $0 < r' \leq r$, $N \in \mathbb{N}$ and a sequence $S_n(r)$ of surfaces such that for all $n \geq N$, $S_n(r)$ separates $B(f^{-n}(x), r)$, $S_n(r) \cap B(f^{-n}(x), r')$ is homotopically trivial within $S_n(r)$, and $S_n(r)$ intersects $U(p, r)$ and they are topologically transversal at their (unique) intersection point.

**2.** Choosing $r > 0$ small enough, we will have, for every convergent subsequence $\{S_{n_k}(2r)\}$ of $\{S_n(2r)\}$, that $H \lim_{k\to\infty} S_{n_k}(2r) \subset S(p, 2r)$. Otherwise we will have points of $W^s_{\epsilon'}(p)$ not in $S(p, 2r)$ for arbitrarily small $\epsilon'$, $0 < \epsilon' \leq \epsilon$.

**3.** In view of 2, let us assume that $\{S_n(2r)\}$ converges in the Hausdorff metric and that $S_n(2r) \subset [S(p, 2r)]_\sigma$ for all $n \in \mathbb{N}$ where $\sigma > 0$ is so small that (see 2.8) for every point $z$ between $S(p, 2r)$ and $S_n(2r)$ in $B(p, 3r/2)$ there is a continuum $C(z) \subset W^u_\epsilon(z)$ which intersects either $S(p, 2r)$ or $S_n(2r)$ (or both).

We assume without loss of generality that $p$ is a hyperbolic fixed point and $x$ is homoclinic, $x \in W^s(p) \cap W^u(p)$.

**Lemma 5.1.** *There is $r'' > 0$ such that for every $z$ between $S(p, 2r)$ and $S_n(2r)$ in $B(p, r'')$ there is a compact connected set $D(z) \subset W^s_\epsilon(z)$, $z \in D(z)$, which separates $B(p, r)$ and $p$ and $f^{-n}(x)$ are in different connected components of $B(p, r)\backslash D(z)$.*

**Proof.** Let $z$ be in the hypotheses of the lemma. As $\Omega(f) = M$, there are sequences $z_j \to z$ and $n_j \to +\infty$ such that $\lim_{j\to+\infty} f^{-n_j}(z_j) = z$. By 3, for every $y$ between $S(p, 2r)$ and $S_n(2r)$ in $B(p, 3r/2)$ there is a continuum $C(y) \subset W^u_\epsilon(y)$ which intersects either $S(p, 2r)$ or $S_n(2r)$. If we have that $C(y)$ intersects only $S(p, 2r)$, then there is a neighborhood $V(y)$

between $S(p,2r)$ and $S_n(2r)$ in $B(p,3r/2)$ such that every $y' \in V(y)$ has a continuum $C(y') \subset W^u_\epsilon(y')$ intersecting $S(p,2r)$ too; otherwise there would be a sequence $y_n \to y$ such that the corresponding $C(y_n)$ intersects only $S_n(2r)$ and taking a convergent subsequence of $\{C(y_n)\}$ we will have (see 2.13 and also [**Vi2**], section 1) a continuum $C(y) \subset W^u_\epsilon(y)$ intersecting $S_n(2r)$.

Therefore we may assume that for the sequence $\{z_j\}$, $C(z_j)$ intersects the same surface, say $S(p,2r)$, in a point $w_j$. Thus $f^{-n_j}(C(z_j))$ intersects $f^{-n_j}(S(p,2r))$ in $f^{-n_j}(w_j)$. By 2.10, given $\lambda > 0$ there is $L \in \mathbb{N}$ such that for all $n \geq L$, for all $w \in M$, $f^{-n}(W^u_\epsilon(w)) \subset W^u_\lambda(f^{-n}(w))$.

Therefore the diameter $diam(f^{-n_j}(C(z))) \to 0$ uniformly with $j \to +\infty$, and, as $f^{-n_j}(z_j) \to z$, we have $f^{-n_j}(w_j) \to z$ too. As in 4.1, see also [**Vi1**], section 2, e have that there is $N \in \mathbb{N}$ and a sequence of surfaces $S_j \subset W^s_\epsilon(f^{-n_j}(w_j))$, such that $S_j$ separates $B(f^{-n_j}(w_j),2r)$ if $n_j \geq N$. To see this observe that, as $M$ is compact, we have that $\{w_j\}$ converges to a point $w$. For, if for a certain subsequence $\{w_{j_k}\}$ we have that $\lim_{k\to+\infty} w_{j_k} = w$ then we have that $\{w\} = C(z) \cap S(p,2r)$ by uniqueness of the intersection point (expansivity of $f$). Take a 2-disk $D \subset S(p,2r)$ such that $w \in int(D)$ and $D \subset W^s_{\epsilon/4}(w)$. Therefore, there is $J \in \mathbb{N}$ such that for all $j \geq J$, $w_j \in int(D)$ and $D \subset W^s_{\epsilon/2}(w_j)$.

Without loss of generality we may assume that $w_j \in int(D)$ for all $j \in \mathbb{N}$. Given an arc $\gamma_j \subset D$ joining $w_j$ with $\partial D$ there is $N_j \in \mathbb{N}$ such that $f^{-n}(\gamma_j)$ reaches the boundary of $B(f^{-n}(w_j),r)$ (same arguments as in 4.1) for all $n \geq N_j$ for all arc $\gamma_j$ (compactness of $D$). It follows, by compactness of $\{w_j\}_{j\in\mathbb{N}} \cup \{w\}$, that there is $N \in \mathbb{N}$ such that for all arc $\gamma$ joining $\partial D$ with a point $w_j$, $f^{-n}(\gamma)$ reaches $\partial B(f_{-n_j}(w_j),2r)$ provided that $n \geq N$. Hence, if $n_j \geq N$, $f^{-n_j}(\gamma)$ reaches $\partial B(f^{-n_j}(w_j),2r))$. From this, as in 2.11, we may prove that the connected component $S_j$ of $W^s_\epsilon(f^{-n_j}(w_j)) \cap B(f^{-n_j}(w_j),2r)$ containing $f^{-n_j}(w_j)$ separates $B(f^{-n_j}(w_j),2r)$. As in 2.13, taking a convergent subsequence from $\{S_j\}$, we have that there is a continuum $D(z) \subset W^s_\epsilon(z)$, $z \in D(z)$. Assume, without loss of generality, that $D(z) = H\lim_{j\to\infty}(S_j)$.

We have that $D(z)$ separates $B(z,2r)$ if $z$ is sufficiently close to $p$, say $z \in B(p, r'')$. For, with similar arguments as those of section 4, we may prove that $S_j$ intersects $U(p, 2r)$ in a single point if we take $z$ near $p$, say $z \in V(p)$. Moreover, $S_j$ separates $U(p, 2r)$ (in fact they are topologically transversal at their intersection point). Then $D(z)$ will intersect $U(p, 2r)$ and as in 2.13 (see also [**Vi2**] Proposition 1.5), we may prove that it separates $U(p, 2r)$ in $B(z, 2r)$. Therefore it separates $B(z, r')$ for $0 < r' \leq 2r$.

Taking $r''$ such that if $z \in B(p, r'')$ then $B(z, r/2) \subset B(p, r) \subset B(z, 2r)$ we may prove that $D(z)$ separates $B(p, r)$. It is clear that $p$ and $f^{-n}(x)$ are in different connected components of $B(p, r) \backslash D(z)$. This finishes the proof. □

**Lemma 5.2.** *For every $z$ between $S(p, r)$ and $S_n(r)$ in $B(p, r'')$, $r''$ as in 5.1, there is a continuum $C(z) \subset W^u_\epsilon(z)$, $z \in C(z)$, such that $C(z)$ is separated by $D(z)$ in $B(p, r)$. Moreover, $C(z)$ joins $S(p, r)$ with $S_n(r)$.*

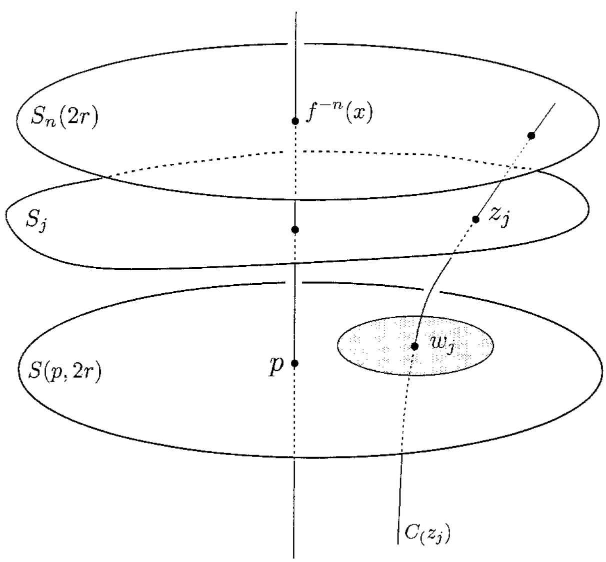


**Figure 4**

**Proof.** By 5.1, for all $z \in B(p, r'')$ we have that $D(z)$ intersects $U(p, r)$. As $\Omega(f) = M$ we have that there are sequences $\{z_j\}$,$\lim_{j \to \infty} z_j = z$, $n_j \to +\infty$ such that $\lim_{j \to \infty} f^{n_j}(z_j) = z$. With arguments similar to those of 5.1 we find $C(z) \subset W^u_\epsilon(z)$ as the Hausdorff limit of a sequence of arcs $\gamma_j \subset W^u_\epsilon(f^{n_j}(w_j))$, where $\{w_j\} = D(z_j) \cap U(p,r)$. Moreover $D(z_j)$ separates $U(p,r)$ into two arcs $U^+(p,r)$ and $U^-(p,r)$ and from this we have two sequences $\{\gamma_j^+\}$ and $\{\gamma_j^-\}$, with $\gamma_j = \gamma_j^+ \cup \gamma_j^-$, $f^{n_j}(w_j) = \gamma_j^+ \cap \gamma_j^-$, $\gamma_j^+ \subset W^u_\epsilon(f^{n_j}(w_j)) \cap f^{n_j}(U^+(p,r/2))$ and $\gamma_j^- \subset W^u_\epsilon(f^{n_j}(w_j)) \cap f^{n_j}(U^-(p,r/2))$ such that both $\gamma_j^+$ and $\gamma_j^-$ reach $\partial B(f^{n_j}(w_j),r)$ and $D(f^{n_j}(w_j))$ locally separates one from each other at $f^{n_j}(w_j)$. As $S_n(r) \subset [S(p,r)]_\sigma$ we have that $\gamma_j^+$ and $\gamma_j^-$ must intersect $S_n(r)$ or $S(p,r)$. But if $\gamma_j^+$ intersects $S(p,r)$ then $\gamma_j^-$ intersects $S_n(r)$ and viceversa; otherwise we violate expansivity.

Thus $C(z)$, being the Hausdorff limit of $\{\gamma_j\}$, intersects both $S_n(r)$ and $S(p,r)$; therefore $C(z)$ joins $S(p,r)$ with $S_n(r)$. We have that $S_n(r) \cap S(p,r) = \emptyset$ and, by 5.1, $D(z)$ separates $U(p,r)$ so it separates $S_n(r)$ from $S(p,r)$. It follows that the points given by $C(z) \cap S(p,r)$ and $C(z) \cap S_n(r)$ are in different components of $B(p,r)$ with respect to $D(z)$. □

**Theorem 5.3.** *There is an open set $W \subset M$ in which there is defined an $f$-lps.*

**Proof.** Let $x$ be a homoclinic point, $r''$ be as in 5.1, $n \in \mathbb{N}$ like in 3, at the begining of this section, and $n' > n$ be such that $f^{-n'}(x) \in B(p,r'')$. Therefore $S_{n'}(r)$ is between $S_n(r)$ and $S(p,r)$. Consider an arc $\gamma$ of points of $U(p,r)$ between $S_n(r)$ and $S(p,r)$ such that $f^{-n'}(x) \in int(\gamma)$ and every point of $\gamma$ is in $B(p,r'')$. Let $D$ be a 2-disk in $S_{n'}(r)$ such that $f^{-n'}(x) \in int(D)$ and $D \subset B(p,r'')$. For every point $y \in D$ there is $C(y)$, as in 5.2, joining $S(p,r)$ with $S_n(r)$; and for every point $z \in \gamma$ there is $D(z)$, as in 5.1, such that $D(z)$ separates $p$ from $f^{-n}(x)$ in $B(p,r)$. Therefore $C(y)$ intersects $D(z)$ and, by the expansive properties of $f$, $C(y) \cap D(z)$ is a single point $w$. Let us define a function $h\colon D \times \gamma \to M$ by $h(y,z) = w$, it is clear that $h(f^{-n'}(x), f^{-n'}(x)) = f^{-n'}(x)$. We claim that $h$ is continuous and injective. Therefore – as $\dim(M) = 3$, $int(D) \times int(\gamma) \simeq \mathbb{R}^3$ and by Brouwer's Theorem of Invariance of Domain (see [**Sp**] Ch. 4) – $h$ is a homeomorphism and its image contains a neighborhood $W$ of $f^{-n'}(x)$. Therefore $h$ defines an $f$-lps in $W$.

Proof of our claim. To see that $h$ is continuous we consider a sequence

$\{(y_n, z_n)\} \subset D \times \gamma$ such that it converges to $(y,z) \in D \times \gamma$ as $n \to \infty$. As $M$ is compact, the sequence $\{w_{n_k}\}$, say to a point $w_\infty$. But $w_{n_k} = C(y_{n_k}) \cap D(z_{n_k})$ hence $w_\infty \in W^u_\epsilon(y) \cap W^s_\epsilon(z)$. Thus $w_\infty = w = C(y) \cap D(z)$ by expansiveness. Therefore $h$ is continuous. To prove that it is injective let $(y,z)$ and $(y',z')$ be points in $D \times \gamma$ such that $h(y,z) = h(y',z') = w$. This is the same to say that $C(y') \cap D(z') = C(y) \cap D(z)$ and therefore $C(y) \cap D(z') = C(y) \cap D(z)$. As $w \in W^s_\epsilon(z)$ and $w \in W^s_\epsilon(z')$ we have that for all $n \geq 0 \, \text{dist}(f^n(z), f^n(z')) \leq 2\epsilon < \alpha$. As $z$, $z' \in \gamma \subset U(p,r)$ we also have that for all $n \leq 0$, $\text{dist}(f^n(z), f^n(z')) < \alpha$. Therefore, by the expansive properties of $f$, $z = z'$. In a similar way we may prove that $y = y'$. This finishes the proof of the claim and the theorem. □

**Proof of Theorem C.** It follows from Theorem A and Theorem 5.3. □

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